MISSILE AUTOPILOT DESIGN USING CONTRACTION THEORY-BASED OUTPUT FEEDBACK CONTROL

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Abstract

This paper addresses missile autopilot design using backstepping controller and nonlinear contracting observer based on contraction theory (also called as contraction analysis). For nonlinear systems, state-feedback controller and stable observer just don't guarantee the stability of combined system. In this paper, under some practical assumption, constraints for design parameters of controller and observer are obtained for exponential stability. Numerical simulations for short-period mode of missile dynamics are carried out to demonstrate the performance and stability.

1 Introduction

Until now, many approaches for nonlinear autopilot design have been proposed[1-3]. For the tail-controlled skid-to-turn missiles, almost nonlinear missile control is applied to control angle-of-attack or sideslip angle which are generally not measureable. However, state-feedback controller and stable observer just don't guarantee the stability of combined system[4-5]. Recently, contraction theory, which is one of incremental form of stability, have been applied to design observer and controller[6-11]. And a systematic method for the design of observer-controller in cascade have been proposed for output feedback[12]. In this paper, we propose a missile autopilot design using contraction-based output feedback control. Under the practical assumption, relationship for design parameters between backstepping controller and nonlinear contracting observer is obtained. After a brief review of contraction theory-based output feedback design in Section 2, Section 3 discusses the application to missile autopilot design. Simulation results are shown in Section 4 and conclusion is given in Section 5

2 Contraction Theory

2.1 Contraction Analysis

Consider the general nonlinear differential equations of the form

$$\dot{x} = f\left(x,t\right) \tag{1}$$

where x is the $n \times 1$ state vector and f is an $n \times 1$ nonlinear vector function. Assuming that $f\left(x,t\right)$ is continuously differentiable, virtual dynamics are introduced by exact differential relation

$$\delta\dot{x} = \frac{\partial f}{\partial x}\left(x,t\right)\delta x \tag{2}$$

where δx is a virtual displacement at fixed time. Consider the local transformation from δx to new δz using a differential coordinate transformation matrix $\Theta\left(x,t\right)$.

$$\delta z = \Theta\left(x,t\right)\delta x \tag{3}$$

Then, the virtual dynamics of δz is represented as equation (4).

$$\delta\dot{z} = F(x,t)\delta z = \left(\dot{\Theta} + \Theta\frac{\partial f}{\partial x}\right)\Theta^{-1}\delta z \qquad (4)$$

where $F(x,t)$ is the generalized Jacobian which represents the covariant derivatives of f in δz coordinates. From the ref [6], the important definition and theorem used in this paper are described hereafter.

Definition 2.1 *Given the system equation (1), a region of the state space is called a contraction region if the Jacobian is uniformly negative definite in that region.*

In the definition, uniformly negative definite means that

$$\exists\beta > 0, \forall x, \forall t \geq 0, \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}^T\right) \leq -\beta I < 0 \qquad (5)$$

Definition 2.2 *Given the system equation (1), a region of the state space is called a contraction region with respect to a uniformly positive definite metric* $M(x,t) = \Theta(x,t)^T\Theta(x,t)$ *, if equivalently* $F(x,t)$ *or* $\frac{\partial f}{\partial x}^T M + \dot{M} + M\frac{\partial f}{\partial x}$ *are uniformly negative definite.*

Theorem 2.1 *Given the system equation (1), any trajectory, which starts in a ball of constant radius with respect to the metric* $M(x,t)$ *, centered at a given trajectory and contained at all times in a contraction region with respect to* $M(x,t)$ *, remains in that ball and converges exponentially to this trajectory.*

When two systems are connected in feedback combination under possibly different metrics

$$\frac{d}{dt}\begin{pmatrix}\delta z_1 \\ \delta z_2\end{pmatrix} = \begin{pmatrix}F_1 & G \\ -G^T & F_2\end{pmatrix}\begin{pmatrix}\delta z_1 \\ \delta z_2\end{pmatrix} \qquad (6)$$

then, the augmented system is contracting if and only if the separated plants are contracting.

2.2 Backstepping Controller

In this section, contraction theory-based recursive controller design is introduced[10]. Let the dynamics of nonlinear system in a strict feedback form be given as

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1 x_2 \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2 x_3 \\
&\vdots \\
\dot{x}_n &= f_n(x_1, x_2, \cdots, x_n) + u
\end{aligned} \qquad (7)$$

The recursive design of controller is given as

$$\begin{aligned}
u(x,Z,t) = &-f_n(x) - g_{n-1}z_{n-2} - \rho_n(z_{n-1}) \\
&+ \frac{\partial\sigma_{n-1}}{\partial x_1}\dot{x}_1 + \sum_{k=1}^{n-2}\frac{\partial\sigma_{n-1}}{\partial z_k}\dot{z}_k
\end{aligned} \qquad (8)$$

with $Z = [x_1, z_1, \ldots, z_{n-1}]^T$ such that the closed-loop system is contracting with its Jacobian J.

$$J = \begin{bmatrix}
-\frac{\partial\rho_1(x_1)}{\partial x_1} & g_1 & 0 & 0 \\
-g_1 & -\frac{\partial\rho_2(z_1)}{\partial x_1} & g_2 & \ddots \\
0 & -g_2 & \ddots & g_{n-1} \\
0 & \ddots & -g_{n-1} & -\frac{\partial\rho_n(z_{n-1})}{\partial x_1}
\end{bmatrix} \qquad (9)$$

The virtual control functions are designed as follows

$$\begin{aligned}
\sigma_1(x_1) &= \frac{1}{g_1}\{-f_1(x_1) - \rho_1(x_1)\} \\
\sigma_j(Z_j) &= \frac{1}{g_j}\{-f_j(\bar{x}_j) - \rho_j(z_{j-1}) - g_{j-1}z_{j-2} \\
&\quad + \frac{\partial\sigma_{k-1}}{\partial x_1}\dot{x}_1 + \sum_{k=1}^{n-2}\frac{\partial\sigma_{k-1}}{\partial z_k}\dot{z}_k\}
\end{aligned} \qquad (10)$$

2.3 Nonlinear Contracting Observer

Consider the observable and controllable nonlinear system given as

$$\begin{aligned} \dot{x} &= f(x,u,t) \\ y &= h(x,t) \end{aligned} \tag{11}$$

where $x \in R^n, u \in R^p, y \in R^m, t \in R_+$ and the f and h are continuously differentiable functions. A nonlinear state observer

$$\dot{\hat{x}} = f(\hat{x},u,t) + K(t)\left(y - h(\hat{x},t)\right) \tag{12}$$

If the tracking controller is represented as

$$u = \phi\left(\hat{x}, x_d(t)\right) \tag{13}$$

then, the closed-loop dynamics become

$$\begin{aligned} \dot{x} &= f\left(x, \phi\left(\hat{x}, x_d(t)\right), t\right) \\ \dot{\hat{x}} &= f\left(\hat{x}, \phi\left(\hat{x}, x_d(t)\right), t\right) + K(t)\left(y - h(\hat{x},t)\right) \end{aligned} \tag{14}$$

Then, the error dynamics of the nonlinear observer is given as

$$\begin{aligned} \dot{\tilde{x}} = {} & f\left(\hat{x}, \phi\left(\hat{x}, x_d(t)\right), t\right) - f\left(\hat{x}, \phi\left(\hat{x}, x_d(t)\right), t\right) \\ & - K(t)\left(h(x,t) - h(\hat{x},t)\right) \end{aligned} \tag{15}$$

The following theorem states the condition for nonlinear contracting observer[12].

Theorem 2.2 *The error dynamics in (15) is uniform global exponential stable if the equation (16) is uniform negative definite.*

$$\frac{\partial f\left(p, \phi\left(p, x_d(t)\right), t\right)}{\partial p} - K(t)\frac{\partial h(p,t)}{\partial p} \tag{16}$$

2.4 Stability of Output Feedback Controller

For the output feedback control, observer and controller should be combined to guarantee the global exponential stability. In this section, we present the stability of contracting observer-based controller[12]. Assuming the desired state $x_d(t)$ is a smooth continuously differentiable function, the error dynamics is given as

$$\dot{e} = \dot{x} - \dot{x}_d = a(e,t) \tag{17}$$

From the design of controller, we can obtain a Lyapunov function of the controller is given as

$$\begin{aligned} & c_1|e|^2 \le V_{con}(e,t) \le c_2|e|^2 \\ & \frac{\partial V_{con}}{\partial t} + \frac{\partial V_{con}}{\partial e} a(e,t) \le -c_3|e|^2 \\ & \left|\frac{\partial V_{con}(e,t)}{\partial e}\right| \le c_4|e| \end{aligned} \tag{18}$$

where c_1, c_2, c_3, c_4 are positive constants. If the full state feedback controller is replaced by the observer-based controller in (13), the stability of the output feedback controller is determined by the following theorem[12-13].

Theorem 2.3 *Assuming the nonlinear observer is contracting, the error dynamics of the output feedback controller is rewritten as*

$$\dot{e} = \dot{x} - \dot{x}_d = a(e,t) + b(e,\tilde{x},t) \tag{19}$$

where the perturbation $b(e,\tilde{x},t)$ satisfies

$$|b(e,\tilde{x},t)| \le \gamma_1|\tilde{x}| + \gamma_2|e|, \quad \forall e, \tilde{x}, t \tag{20}$$

where $|\bullet|$ denotes the Euclidian norm and γ_1, γ_2 are two non-negative constants satisfying

$$\gamma_1^2 < \frac{4(c_3 - c_4\gamma_2)\lambda_q}{c_4^2}, \gamma_2 < \frac{c_3}{c_4^2} \tag{21}$$

where λ_q is obtained from

$$V_{obs} = \tilde{x}^T\tilde{x}, \quad \dot{V}_{obs} \le -\lambda_q \tilde{x}^T\tilde{x} \tag{22}$$

then, the equilibrim point $(\tilde{x}, e) = (0,0)$ is uniform global exponential stable.

3 The dynamic model of STT Missile

In general, tail-controlled skid-to-turn missiles with cruciform configuration, pitch and yaw autopilot are identically designed assuming small cross-coupling effect among roll, pitch, and yaw channels. Therefore, the design of pitch autopilot is only dealt with in this paper. Under the assumption that the missile is rigid and the gravitational force is ignored, a nonlinear short-period dynamic model is given by the equation (23)~(25).

$$\dot{\alpha} = q - K_1 C_{N_0}(\alpha) - K_1 C_{N_\delta}\delta \tag{23}$$

$$\dot{q} = K_2\left(C_{M_0}(\alpha) + K_3 C_{M_q} q + C_{M_\delta}\delta\right) \tag{24}$$

$$a_n = VK_1\left(C_{N_0}(\alpha) + C_{N_\delta}\delta\right) \tag{25}$$

where $K_1 = QS/(mV), K_2 = QSD/I_{yy}, K_3 = D/(2V)$ and α, q, a_n, δ represents angle-of-attack, pitch rate, normal acceleration, fin deflection angle, respectively. m, I_{yy} mean mass and moment of inertia and V, D, S, Q describe missile velocity, reference length, reference area, dynamic pressure. The aerodynamic coefficients are generally obtained from wind tunnel test or semi-empirical code in the tabular form. In this paper, C_{N_0} and C_{M_0} are approximated as fourth-order polynomial equations of angle-of-attack to describe smooth nonlinear system.

$$\begin{aligned} C_{N_0} &= a_4\alpha^4 + a_3\alpha^3 + a_2\alpha^2 + a_1\alpha + a_0 \\ C_{M_0} &= b_4\alpha^4 + b_3\alpha^3 + b_2\alpha^2 + b_1\alpha + b_0 \end{aligned} \tag{26}$$

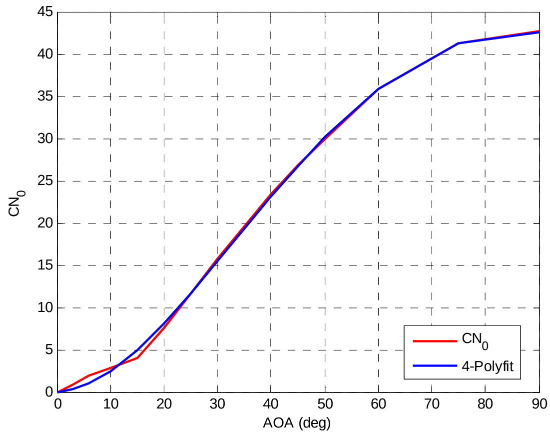

Figure 1. Approximation of C_{N_0}

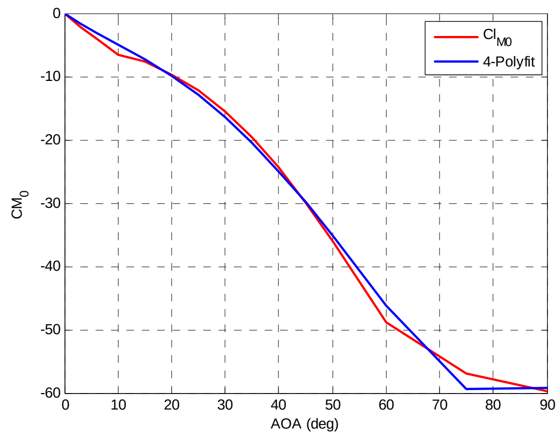

Figure 2. Approximation of C_{M_0}

In this paper, C_{N_δ} and C_{M_δ} are assumed to be only a function of Mach number because the variation of angle-of-attack in the flight regime doesn't have much effect on them comparing to C_{N_0} and C_{M_0}. Figure 1 and Figure 2 represents the approximation results for normal force and pitch moment coefficients at fixed Mach number, respectively.

4 Autopilot Design

4.1 Controller Design

In the equation (23), $K_1 C_{N_\delta}\delta$, the normal force due to deflection angle, is generally negligible because the most normal force is induced by term $K_1 C_{N_0}(\alpha)$. Therefore, the equations (23) and (24) are reformulated in a strict feedback form as follows :

$$\dot{\alpha} = -K_1 C_{N_0}(\alpha) + q \tag{27}$$

$$\dot{q} = K_2\left(C_{M_0}(\alpha) + K_3 C_{M_q} q\right) + K_2 C_{M_\delta}\delta \tag{28}$$

According to the Section 2.2, let new variables be given as

$$e_1 = \alpha - \alpha_c, e_2 = q - q_d \tag{29}$$

The error dynamics of angle-of-attack is represented as follows :

$$\begin{aligned}\dot{e}_1 &= -K_1 C_{N_0}(\alpha) + q - \dot{\alpha}_c \\ &= -K_1 C_{N_0}(\alpha) + (e_2 + q_d) - \dot{\alpha}_c\end{aligned} \tag{30}$$

If we let

$$q_d = -\lambda_1 e_1 + K_1 C_{N_0}(\alpha) + \dot{\alpha}_c \tag{31}$$

the error dynamics of angle-of-attack becomes

$$\dot{e}_1 = -\lambda_1 e_1 + e_2 \tag{32}$$

And e_2 dynamics is described as follows :

$$\begin{aligned}\dot{e}_2 = \dot{q} - \dot{q}_d = K_2\left(C_{M_0}(\alpha) + K_3 C_{M_q} q\right) \\ + K_2 C_{M_\delta}\delta - \dot{q}_d\end{aligned} \tag{33}$$

If we let

$$\delta = \frac{1}{K_2 C_{M_\delta}}\left\{-\lambda_2 e_2 - e_1 - K_2\left(C_{M_0}(\alpha) + K_3 C_{M_q} q\right) + \dot{q}_d\right\} \tag{34}$$

the error dynamics of pitch rate becomes

$$\dot{e}_2 = -e_1 - \lambda_2 e_2 \tag{35}$$

The deflection angle is rewritten as the following equation through $\dot{q}_d$ calculation.

$$\begin{aligned}\delta &= \frac{1}{K_2 C_{M_\delta}}\left\{-\lambda_2 e_2 - e_1 - K_2\left(C_{M_0}(\alpha) + K_3 C_{M_q} q\right)\right\} \\ &+ \frac{1}{K_2 C_{M_\delta}}\left\{\begin{aligned}&-\lambda_1 e_2 + \lambda_1^2 e_1 - K_1^2 \frac{\partial C_{N_0}(\alpha)}{\partial \alpha} C_{N_0}(\alpha) \\ &+ K_1 \frac{\partial C_{N_0}(\alpha)}{\partial \alpha} q + \ddot{\alpha}_d\end{aligned}\right\} \\ &= \frac{1}{K_2 C_{M_\delta}}\left\{\begin{aligned}&-(\lambda_1 + \lambda_2) e_2 + \left(\lambda_1^2 - 1\right) e_1 + \ddot{\alpha}_d \\ &+ \left(K_1 \frac{\partial C_{N_0}(\alpha)}{\partial \alpha} - K_2 K_3 C_{M_q}\right) q \\ &- K_1^2 \frac{\partial C_{N_0}(\alpha)}{\partial \alpha} C_{N_0}(\alpha) - K_2 C_{M_0}(\alpha)\end{aligned}\right\}\end{aligned} \tag{36}$$

The derived error dynamics in (32) and (35) can be written in differential framework as

$$\begin{bmatrix}\delta\dot{e}_1 \\ \delta\dot{e}_2\end{bmatrix} = \begin{bmatrix}-\lambda_1 & 1 \\ -1 & -\lambda_2\end{bmatrix}\begin{bmatrix}\delta\dot{e}_1 \\ \delta\dot{e}_2\end{bmatrix} \tag{37}$$

Therefore, for some positive λ_1, λ_2, the state feedback controller is stable. A Lyapunov function of the controller is determined as

$$\begin{aligned}&V_{con}(e,t) = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2 \\ &\frac{dV_{con}}{dt}(e,t) = e_1\dot{e}_1 + e_1\dot{e}_2 \\ &\quad = -\lambda_1 e_1^2 - \lambda_2 e_2^2 \le -\min(\lambda_1, \lambda_2)|e|^2 \\ &\left|\frac{\partial V_{con}(e,t)}{\partial e}\right| = |e|\end{aligned} \tag{38}$$

The equation (38) means that

$$c_1 < 1, c_2 > 1, c_3 = \min(\lambda_1, \lambda_2), c_4 = 1 \tag{39}$$

4.2 Output Feedback Design

For the output feedback control, the estimated state should be used as follows :

$$\delta = \frac{1}{K_2 C_{M_\delta}}\left\{-\lambda_2 \hat{e}_2 - \hat{e}_1 - K_2\left(C_{M_0}(\hat{\alpha}) + K_3 C_{M_q}\hat{q}\right) + \dot{\hat{\sigma}}_1\right\} \tag{40}$$

Let the estimation error be defined as

$$\begin{aligned}\tilde{\alpha} &= \alpha - \hat{\alpha} = e_1 - \hat{e}_1 \\ \tilde{q} &= q - \hat{q} = e_2 - \hat{e}_2 + \left(q_d - \hat{q}_d\right)\end{aligned} \tag{41}$$

Then, error dynamics is rewritten as follows :

$$\begin{aligned}&\dot{e}_1 = -\lambda_1 e_1 + e_2 \\ &\dot{e}_2 = \dot{q} - \dot{q}_d = K_2\left(C_{M_0}(\alpha) + K_3 C_{M_q} q\right) \\ &\qquad + K_2 C_{M_\delta}\delta - \dot{q}_d \\ &\quad = -e_1 - \lambda_2 e_2 + b_2(e, \tilde{x}, t)\end{aligned} \tag{42}$$

where $b_2(e,\tilde{x},t)$ is the perturbed term caused by the estimated error which is given as

$$\begin{aligned} b_2(e,\tilde{x},t) &= \left(\lambda_2 + K_2K_3C_{M_q}\right)\tilde{q} + \tilde{\alpha} \\ &\quad + K_2\left(C_{M_0}(\alpha) - C_{M_0}(\hat{\alpha})\right) \\ &\quad - \lambda_2(q_d - \hat{q}_d) - \left(\dot{q}_d - \dot{\hat{q}}_d\right) \end{aligned} \tag{43}$$

Calculation for $(q_d - \hat{q}_d), \left(\dot{q}_d - \dot{\hat{q}}_d\right)$ is omitted due to the limited space. Assuming $\tilde{\alpha}$ is small and $\partial C_{M_0}(\alpha)/\partial\alpha$ and $\partial C_{N_0}(\alpha)/\partial\alpha$ are comparatively constant, then

$$\begin{aligned} &\left(C_{M_0}(\alpha) - C_{M_0}(\hat{\alpha})\right) \approx \frac{\partial C_{M_0}}{\partial\alpha}(\hat{\alpha})\tilde{\alpha} \\ &\left(C_{N_0}(\alpha) - C_{N_0}(\hat{\alpha})\right) \approx \frac{\partial C_{N_0}}{\partial\alpha}(\hat{\alpha})\tilde{\alpha} \\ &\left(\frac{\partial C_{N_0}}{\partial\alpha}(\alpha)q - \frac{\partial C_{N_0}}{\partial\alpha}(\hat{\alpha})\hat{q}\right) \\ &\qquad \approx \frac{\partial C_{N_0}}{\partial\alpha}(\hat{\alpha})\tilde{q} + \frac{\partial^2 C_{N_0}}{\partial\alpha^2}(\hat{\alpha})\hat{q}\tilde{\alpha} \end{aligned} \tag{44}$$

Then, coefficients of $\tilde{\alpha}$ including K_1 terms are comparatively small, the $b_2(e,\tilde{x},t)$ becomes

$$\begin{aligned} b_2(e,\tilde{x},t) &= \left(\lambda_1 + \lambda_2 + K_2K_3C_{M_q} - K_1\frac{\partial C_{N_0}}{\partial\alpha}(\hat{\alpha})\right)\tilde{q} \\ &\quad + (1+\lambda_1\lambda_2)\tilde{\alpha} \\ &\quad + K_2\left(C_{M_0}(\alpha) - C_{M_0}(\hat{\alpha})\right) \\ &\quad - K_1(\lambda_1+\lambda_2)\left(C_{N_0}(\alpha) - C_{N_0}(\hat{\alpha})\right) \end{aligned} \tag{45}$$

Then,

$$b(e,\tilde{x},t) = \begin{bmatrix} 0 & 0 \\ \left(\begin{array}{l} 1+\lambda_1\lambda_2 + K_2\dfrac{\partial C_{M_0}}{\partial\alpha}(\hat{\alpha}) \\ -K_1(\lambda_1+\lambda_2)\dfrac{\partial C_{N_0}}{\partial\alpha}(\hat{\alpha}) \end{array}\right) & \left(\begin{array}{l} (\lambda_1+\lambda_2) \\ +K_2K_3C_{M_q} \\ -K_1\dfrac{\partial C_{N_0}}{\partial\alpha}(\hat{\alpha}) \end{array}\right) \end{bmatrix} \begin{bmatrix} \tilde{\alpha} \\ \tilde{q} \end{bmatrix} \tag{46}$$

Therefore, γ_1, γ_2 in (20) are determined as follows

$$\begin{aligned} \gamma_1 &= \left\| \begin{bmatrix} 0 & 0 \\ \left(\begin{array}{l} 1+\lambda_1\lambda_2 + K_2\dfrac{\partial C_{M_0}}{\partial\alpha}(\hat{\alpha}) \\ -K_1(\lambda_1+\lambda_2)\dfrac{\partial C_{N_0}}{\partial\alpha}(\hat{\alpha}) \end{array}\right) & \left(\begin{array}{l} (\lambda_1+\lambda_2) \\ +K_2K_3C_{M_q} \\ -K_1\dfrac{\partial C_{N_0}}{\partial\alpha}(\hat{\alpha}) \end{array}\right) \end{bmatrix} \right\|_F \\ \gamma_2 &= 0 \end{aligned} \tag{47}$$

where F means Frobenius norm. From the equation (21), we can obtain a constraint for controller design parameters, λ_1, λ_2.

$$\gamma_1^2 < \frac{4(c_3 - c_4\gamma_2)\lambda_q}{c_4^2} = 4\cdot\min(\lambda_1,\lambda_2)\cdot\lambda_q \tag{48}$$

4.3 Observer Design

A nonlinear contracting observer is designed using the system dynamics.

$$\begin{aligned} \dot{\hat{\alpha}} &= -K_1C_{N_0}(\hat{\alpha}) + \hat{q} \\ &\quad + \beta_1(a_n - \hat{a}_n) + \beta_2(q - \hat{q}) \end{aligned} \tag{49}$$

$$\begin{aligned} \dot{\hat{q}} &= K_2\left(C_{M_0}(\hat{\alpha}) + K_3C_{M_q}\hat{q}\right) + K_2C_{M_\delta}\delta \\ &\quad + \beta_3(q - \hat{q}) \end{aligned} \tag{50}$$

The total Jacobian for contracting observer should be uniform negative definite and consists of observer part and controller part.

$$F_{total} = F_{obs} + F_{con} \tag{51}$$

If there is no input, the Jacobian for the observer system with respect to $(\hat{\alpha}, \hat{q})$ is described as

$$F_{obs} = \begin{bmatrix} -K_1\dfrac{\partial C_{N_0}}{\partial\alpha}(\hat{\alpha})(1+\beta_1V_M) & 1-\beta_2 \\ K_2\dfrac{\partial C_{M_0}}{\partial\alpha}(\hat{\alpha}) & K_2K_3C_{M_q} - \beta_3 \end{bmatrix} \tag{52}$$

The additional contribution of the output feedback controller is given as

$$F_{con} = \begin{bmatrix} 0 & 0 \\ K_2 C_{M_\delta} \dfrac{\partial \delta}{\partial \hat{\alpha}} & K_2 C_{M_\delta} \dfrac{\partial \delta}{\partial \hat{q}} \end{bmatrix} \tag{53}$$

where

$$\frac{\partial \delta}{\partial \hat{\alpha}} = \frac{1}{K_2 C_{M_\delta}} \left\{ \begin{array}{l} K_1(\lambda_1+\lambda_2)\dfrac{\partial C_{N_0}}{\partial \alpha}(\hat{\alpha}) - (1+\lambda_1\lambda_2) \\ +K_1 \dfrac{\partial^2 C_{N_0}}{\partial \alpha^2}(\hat{\alpha})\hat{q} - K_2 \dfrac{\partial C_{M_0}}{\partial \alpha}(\hat{\alpha}) \\ -K_1^2 \left(\dfrac{\partial^2 C_{N_0}}{\partial \alpha^2}(\hat{\alpha}) + \left(\dfrac{\partial C_{N_0}}{\partial \alpha}(\hat{\alpha}) \right)^2 \right) \end{array} \right\}$$

$$\frac{\partial \delta}{\partial \hat{q}} = \frac{1}{K_2 C_{M_\delta}} \left\{ \begin{array}{l} -(\lambda_1+\lambda_2) + K_1 \dfrac{\partial C_{N_0}}{\partial \alpha}(\hat{\alpha}) \\ -K_2 K_3 C_{M_q} \end{array} \right\}$$

Finally, the conditions for design parameter of observer and controller are given as

$$\begin{aligned} & -K_1 \frac{\partial C_{N_0}}{\partial \alpha}(\hat{\alpha})(1+\beta_1 V_M) < 0 \\ & \beta_2 = 1 + K_2 \frac{\partial C_{M_0}}{\partial \alpha}(\hat{\alpha}) + K_2 C_{M_\delta} \frac{\partial \delta}{\partial \hat{\alpha}} \\ & K_2 K_3 C_{M_q} - \beta_3 + K_2 C_{M_\delta} \frac{\partial \delta}{\partial \hat{q}} < 0 \\ & \gamma_1^2 < 4 \cdot \min(\lambda_1, \lambda_2) \cdot \lambda_q \end{aligned} \tag{54}$$

where

$$\lambda_q = 2 \times \min \left(\begin{array}{l} K_1 \dfrac{\partial C_{N_0}}{\partial \alpha}(\hat{\alpha})(1+\beta_1 V_M), \\ \beta_3 - K_2 K_3 C_{M_q} - K_2 C_{M_\delta} \dfrac{\partial \delta}{\partial \hat{q}} \end{array} \right)$$

4.4 Simulation Results

A nonlinear model of SRAAM(Short-Range Air-to-Air Missile) is used to design the output feedback controller. The result of short-period mode at Mach 1.2 represents lightly damped characteristics as in Figure 3. The initial values of the states are $(\alpha_0, q_0) = (5^\circ, -10^\circ/s)$

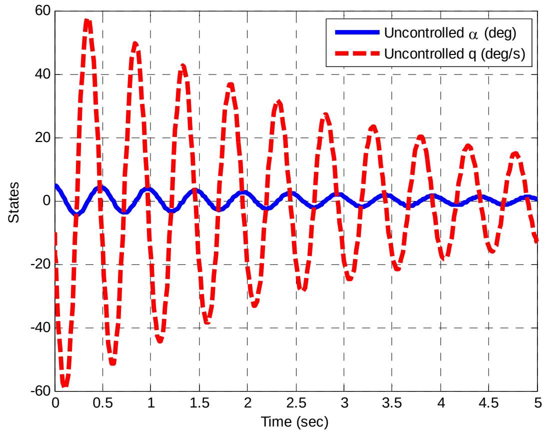

Figure 3. Response of Uncontrolled System

Figure 4 represents the design result for only backstepping controller. In this simulation, we use $\lambda_1 = \lambda_2 = 5$ and $\alpha_c = 3^\circ$. The result shows that the backstepping controller provides a good tracking performance.

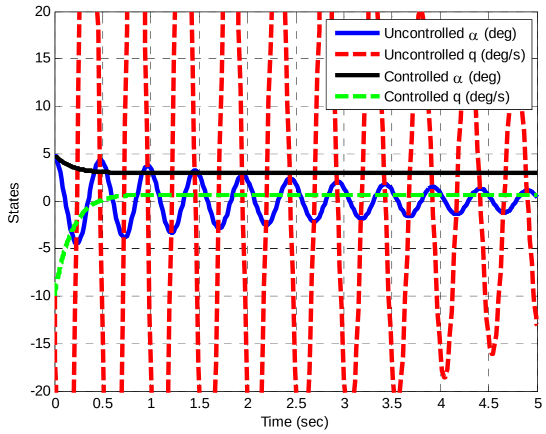

Figure 4. Response of Controlled System

Figure 5 represents the response for observer-based backstepping controller. To satisfy the constraints (54), the design parameters of controller and observers are set as

$$\begin{aligned} & \lambda_1 = \lambda_2 = 12 \\ & \beta_1 = \beta_2 = 1 \end{aligned} \tag{55}$$

In this simulation, the initial estimated states are $(\hat{\alpha}_0, \hat{q}_0) = (10^\circ, 5^\circ / s)$. The result shows that the estimation error of observer sufficiently decreases in 0.2 sec.

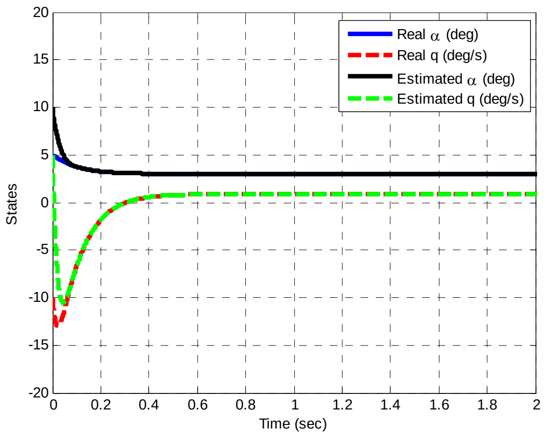

Figure 5. Response of Output Feedback Control without Parameter Constraints

Figure 6 represents that the estimation error in observer and tracking error in controller. It shows that the errors exponentially decreases as expected.

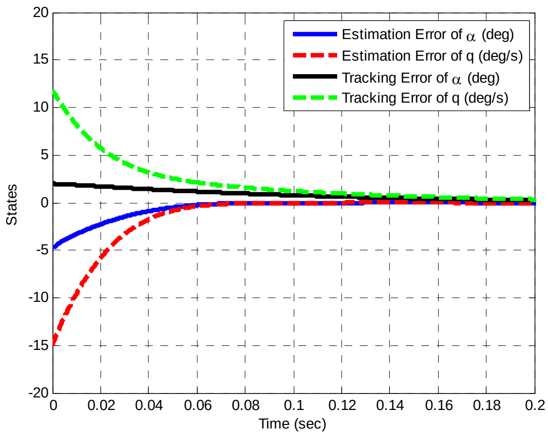

Figure 6. Estimation and Tracking Error

5 Conclusion

In this paper, missile autopilot design using output feedback control is proposed using contraction methodology. To guarantee exponential stability of the output feedback controller, constraints of design parameters for backstepping controller and nonlinear contracting observer are obtained under the practical assumption. The simulation results shows that the proposed stable output feedback controller provide good performance with exponential stability.

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