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RESEARCH ON THE IMPACT OF SHOCK WAVE TO EXTENSIONALITY OF POD METHOD

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Abstract

In this paper, flow reconstruction accuracy and flow prediction capability of discontinuous transonic flow field by means of proper orthogonal decomposition (POD) method is studied. The ability to reconstruct and predict flow flied is called extensionality of POD method in this paper. Although, only using a few modes, we can capture the information of subsonic flow field. When it counters the shock wave, the smoothness of the solution near the shock wave cannot be guaranteed. The modal coefficients are interpolated or extrapolated and different modal components are superposed to realize the prediction of the flow field beyond the snapshot sets. Results show that compared with the subsonic flow, the transonic flow with shock wave requires more POD modes to reach a comparative reconstruction accuracy. When a shock wave exists, the interpolation prediction ability is acceptable. However, large errors exist in extrapolation, and increasing the number of POD modes cannot effectively improve the prediction accuracy of the flow field.

1 Introduction

The proper orthogonal decomposition (POD), also known as K-L expansion or principle components analysis, has been widely used in many areas, such as image processing [1], pattern recognition [2], reduced order model [3] (ROM) as well as steady flow field analysis and airfoil design optimization [4], and so on. POD method is a powerful statistical tool which can extract the significant structure or pattern from a large data set. POD method is also an effective reduction tool which can use the fewest POD modes to present a large data ensemble with the given accuracy. Lumley [5] firstly introduced the POD method into the turbulent flow. Then, Sirovich [6] introduced the snapshots as a way to efficiently determine the POD modes which made POD method applied to broader problems, especially computational fluid dynamics (CFD).

Building mathematical model with POD method is an efficient way to realize the reconstruction and prediction of the flow field. A set of instantaneous flow solutions, or "snapshots" is obtained from a simulation of the CFD method or from experimental data. These snapshots are used to construct an eigenvalue problem. Solving this problem can get a set of POD modes, which is the optimal representation of the flow. The ROM can be derived through projecting the CFD model onto the reduced space spanned by the POD modes. This ROM can predict the modal coefficients at any time, so as to realize the reconstruction and prediction of the flow field. This method has been successfully applied to unsteady aerodynamic problems, for example in Ref [7-10]. Furthermore, Kim [11] developed the ROM in frequency form using a set of discrete snapshots in frequency domain instead of time domain. In addition to constructing ROM by projection, the modal coefficients can also be obtained by the method of system identification. Walton et al. [12] combined POD method with radial basis function (RBF) method to achieve the prediction of the flow field. Apart from the usage of POD to capture the time variation of fluid dynamic problems, it can also be used to capture parametric variation. With the snapshots both in time and over a range of interblade phase angles at a fixed Mach number, Epureanu et al. [13] have developed ROMs for turbomachinery flows using POD method. They applied the ROM to various sets of Mach numbers which were not contained in the snapshots set. The results showed that accurate results were obtained at Mach numbers close to those used in the snapshots set. Eversion and Sirovich[14] developed Gappy POD method on the basis of POD method which can repair damaged data and construct missing or "gappy" data, which has been successfully applied to the characterization of human faces. The Gappy POD has also been applied to the reconstruction of airfoil pressure field from limited surface pressure data and a few POD modes [15].

These studies show that POD method has higher accuracy in the reconstruction of snapshot and has some ability to interpolation in the state space. However, when there is a shock wave in the flow field, whether the reconstruction precision and interpolation capability of POD method will be affected. Thus, this paper conducted a study on this question. the extensionality of POD method are tested in both subsonic flow and transonic flow.

2 POD method and interpolation

2.1 POD procedure

POD can be applied efficiently to large systems using the method of snapshots[6] as follows. $\{U^k\}_{k=1}^{m}$ is a collection of m flow snapshots, where U^k is a vector containing the flow solution at a time or a parameter, such as the angle of attack or Mach number. And usually these solutions are expressed as the sum of average values and fluctuation values.

$$U^k = \bar{U} + \tilde{U}^k \tag{1}$$

The correlation matrix R is formed by computing the inner product between every pair of snapshots,

$$R_{ik} = \left(U^i, U^k\right) \tag{2}$$

where $\left(U^i, U^k\right)$ denotes the inner product between U^i and U^k . And then compute the eigenvalues λ_i and eigenvectors Ψ^i . The orthonormal POD modes can be obtained by the following formula:

$$\left\{\Phi^i\right\}_{i=1}^{m} = \frac{1}{\sqrt{\lambda_i}}\left\{U^i\right\}_{i=1}^{m}\left\{\Psi^i\right\}_{i=1}^{m} \tag{3}$$

The magnitude of the ith eigenvalue, λ_i , describes the relative importance of the ith POD mode, also known as the relative energy contained in the ith POD mode.

The approximate reconstruction of the flow solutions can be given by the sum of average values and a linear combination of the POD modes:

$$U^k \approx \bar{U} + \sum_{i=1}^{p} \alpha_i^k \Phi^i \tag{4}$$

where $p \ll m$ and p is chosen to capture the desired level of energy. α_i^k is the modal coefficient, corresponding to the ith POD mode, which can be obtained by projecting the kth snapshot to the ith POD mode.

$$\alpha_i^k = \left(\Phi^i, U^k\right) \tag{5}$$

2.2 POD with interpolation and extrapolation procedure

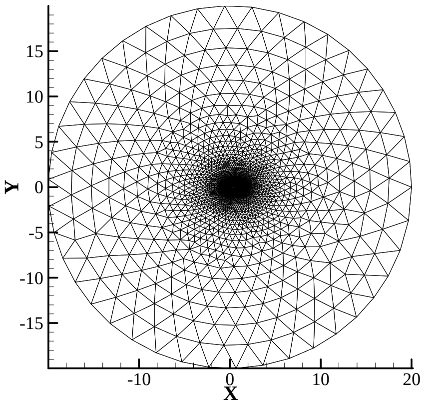
POD method combined with interpolation and extrapolation can realize the fast prediction of flow solutions which are not contained in the snapshots[16]. The main steps are as follows:

1. $\{U^{\delta_k}\}_{k=1}^{m}$ is the set of snapshots varying with time or the flow parameter, which is described by δ.
2. Perform the basic POD procedure described above to get the truncated orthonormal POD modes $\left\{\Phi^i\right\}_{i=1}^{p}$ and the corresponding POD coefficients $\alpha_i^{\delta_k}$.
3. $\left\{\alpha_i^{\delta_k}\right\}_{k=1}^{m}$ is a function of δ , and interpolation or extrapolation can be used to determine the POD coefficients of δ that are not included in the original ensemble. We choose the cubic spline interpolation as the interpolation and extrapolation method in this paper and the prediction flow solution at any δ value is given by

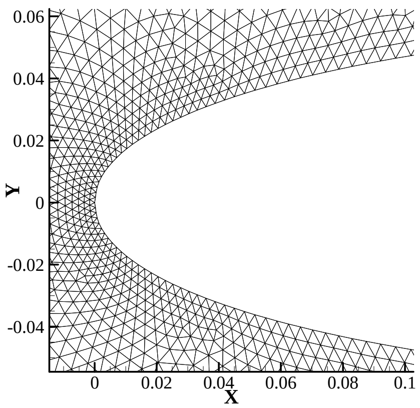
$$U^\delta \approx \bar{U} + \sum_{i=1}^{p} \alpha_i^\delta \Phi^i \tag{6}$$

3 Results and discussion

In this paper, the steady flow solutions to a NACA 0012 airfoil with varying angles of attack are used as snapshots. CFD solver adopts AUSM+UP scheme to solve the Euler equation. Unstructured grid is used. The number of nodes is 6916 and the number of cells is 13490, as shown in figure 1. A detailed description of the solver and its verification can be referred to in Ref [17].



(a) Computational grids



(b) Close-up view of grids
Fig. 1. Computational grids.

3.1 Reconstruction of the flow field

The snapshots set of case 1 is composed of a 100-member steady flow solution ensemble at the fixed Mach number of 0.8, at angles of attack in the range $\left[0.25^{\circ}, 2.23^{\circ}\right]$, uniformly spaced with an interval of 0.02°. Among them, two shock waves are separately located on the upper surface and the lower surface of the airfoil in the first 70 snapshots, while there is only one shock wave on the upper surface in the last 30 snapshots. With the increase of angle of attack, the position of the shock wave gradually moves towards the trailing edge of the airfoil. And the range of the shock wave on the upper surface corresponding to the x-axis is 0.53~0.71. Figure 2 shows the distribution of pressure coefficients on the upper surface at different angles of attack, which is to illustrate the position change of the shock wave at different angles of attack. Based on this snapshots set, reconstruction of the flow solution is conducted by the POD method. For a distinct demonstration, POD method is applied only to the pressure field and the pressure is dimensionless. The procedure of the other flow fields is straightforward.

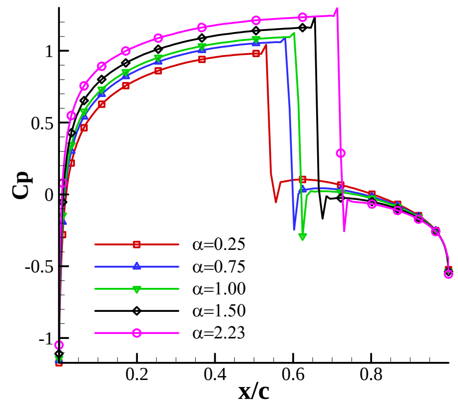


Fig. 2. Pressure coefficient distribution of airfoil at different angles attack

In order to better illustrate the impact of shock wave on the reconstruction results of the flow field based on POD method, a subsonic case 2 is adopted as a contrast. The snapshots set of case 2 is composed of a steady flow solution ensemble at the fixed Mach number of 0.5 and the range of angle of attack, spacing and the number of snapshots are all the same as in case 1. Figure 3 shows the eigenvalue curves of the two cases. It can be seen that the eigenvalue of $M = 0.8$ declines slowly and the phenomenon of “most energy contained in the first few POD modes” is not obvious. As shown in table 1, at $M = 0.5$, it only needs one POD mode to achieve 99.9% of the total energy; while at $M = 0.8$, it requires 9 POD modes to achieve 99% of the total energy and 21 POD modes to achieve 99.9% of the total energy. Therefore, if the energy is regarded as a standard of POD mode truncation, the flow with shock waves requires more POD modes to reach the specified energy. Figure 4 shows the first three POD modes of these two cases. The positive and negative pressure distribution can be seen from the POD modes of case 1. While the POD modes of case 2 is smooth. Taking the 71th snapshot as an example, figure 5 shows the comparison of surface pressure coefficient obtained by POD and CFD in the two cases. Figure 5(a) shows that in the state of the transonic speed with shock waves ($M = 0.8$), even the POD modes which are needed to reach 99.9% of the total energy are used, oscillation phenomenon still occurs before and after the shock wave. But outside this region, even one POD mode is used, surface pressure coefficients obtained by POD agree well with those by CFD. However, in the state of the subsonic speed ($M = 0.5$), with only one POD mode, surface pressure coefficients both at the shock wave and other regions obtained by POD agree well with those by CFD, as shown in figure 5(b). This difference is mainly caused by the shock discontinuity in snapshots.

In order to quantify the deviation between the reconstructed flow field and the actual flow field, the reconstruction error is defined as follows:

$$\text{error} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(P_{POD}^{i} - P_{CFD}^{i}\right)^2} \tag{7}$$

where P_{POD}^{i} and P_{CFD}^{i} are the pressure values of airfoil surface and n represents the number of pressure points on the airfoil surface. Figure 6 shows the reconstruction error curve with the order of POD mode. It can be seen that the reconstruction error of $M = 0.8$ is 2~3 orders of magnitude higher than that of $M = 0.5$ and the curve convergence speed of $M = 0.8$ is relatively slow. But if enough POD modes are adopted, the case of $M = 0.8$ can also reach a higher reconstruction accuracy. Therefore, the existence of the shock discontinuity in the flow leads to the increase of the reconstruction error using POD method.

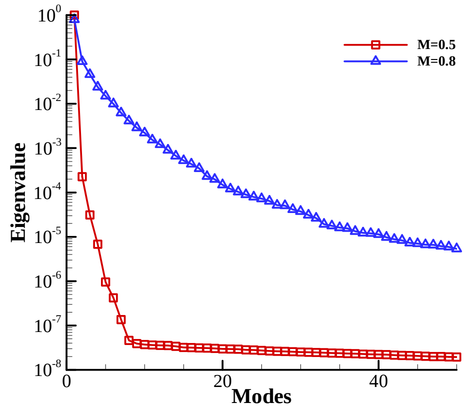
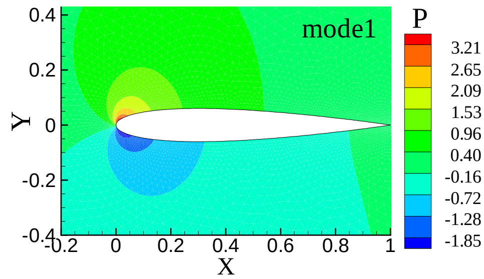


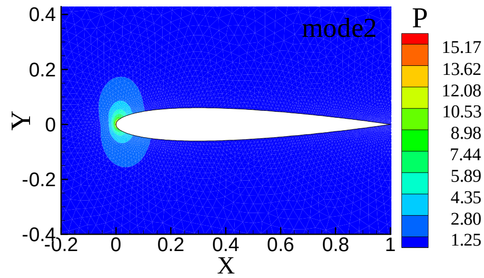
Fig. 3. Eigenvalue curve

Table 1 The order of POD mode needed to achieve specified energy

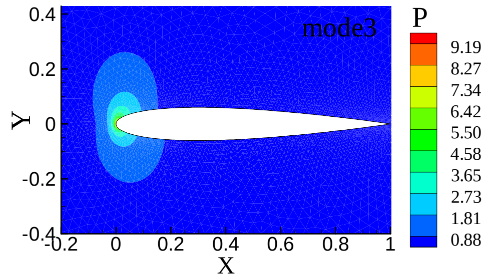
	99%	99.9%
$M = 0.8$	9	21
$M = 0.5$	1	1



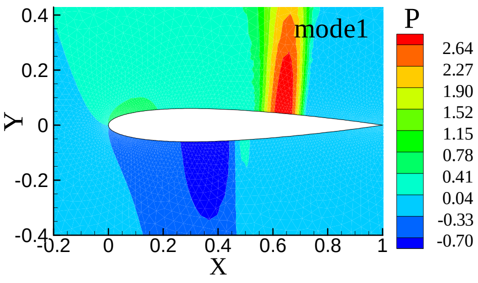
(a)



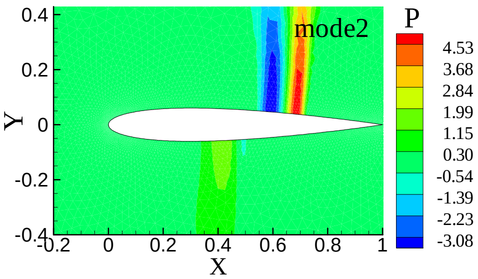
(b)



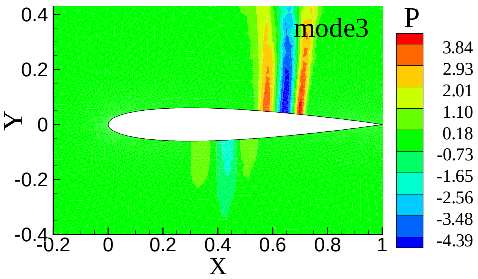
(c)



(d)

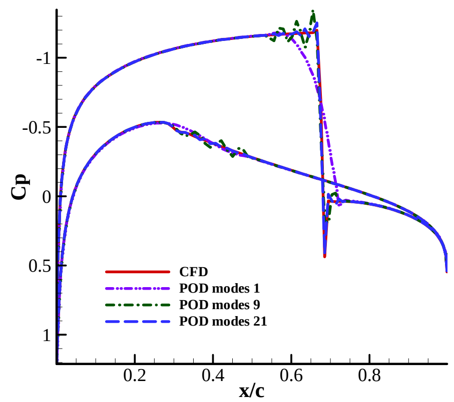


(e)

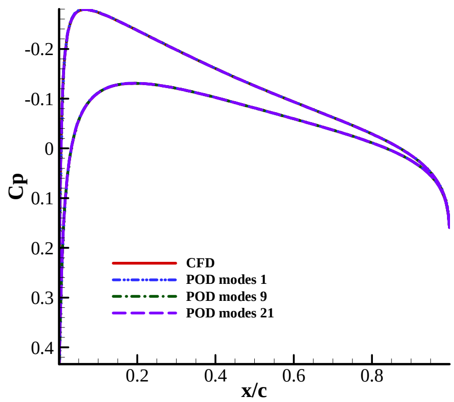


(f)

Fig. 4. POD modes ((a)~(c): case 2; (d)~(f): case 1)



(a) $M = 0.8$



(b) $M = 0.5$

Fig. 5. Comparison of the surface pressure coefficient between the POD and the CFD at snapshot 71($\alpha = 1.65^{\circ}$)

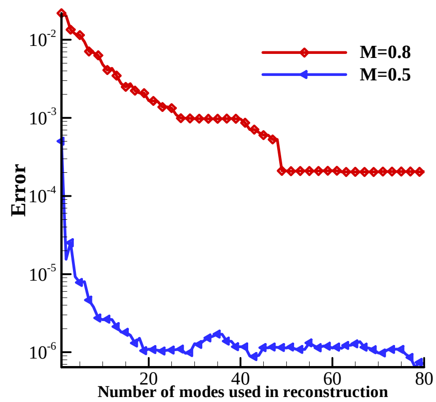


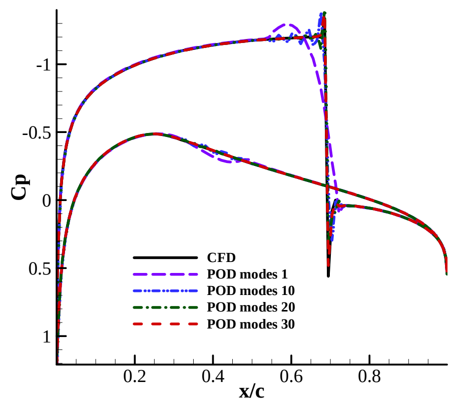
Fig. 6. Reconstruction error curve (snapshot 71, $\alpha = 1.65^{\circ}$)

3.2 Prediction of the flow field

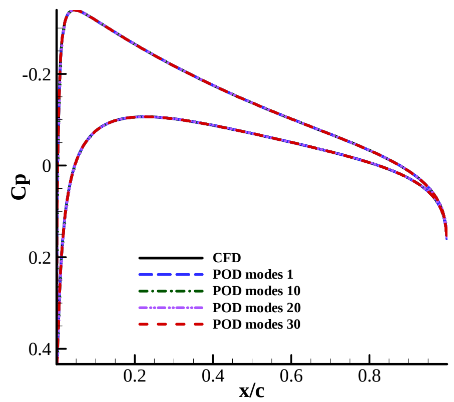
We also take case 1 and case 2 described above as comparison. POD coefficients are interpolated or extrapolated according to the above interpolation steps, so as to realize the prediction of flow field solutions which are not contained in the snapshots set. The flow field of $\alpha = 1.84^{\circ}$ is taken as the presentation example in interpolation and the flow field of $\alpha = 2.26^{\circ}$ in extrapolation.

3.2.1 Interpolation

For flow with the shock discontinuity, POD method combined with interpolation has a relatively good prediction ability. But the shock discontinuity in the flow also causes the increase of the prediction error in POD method combined with interpolation. The analysis results are similar to that of the reconstruction. The surface pressure coefficient curve of POD interpolation at $M = 0.8$ still shows oscillation phenomenon before and after the shock wave while other regions agree very well with that of CFD, as shown in figure 7. Furthermore, with the increase of POD modes, the oscillation frequency increases gradually and the amplitude decreases gradually. And with enough POD modes, it can achieve a higher reconstruction accuracy. Figure 8 shows the interpolation error curve with the POD modes. It can be seen that the interpolation error curves of the two cases are all convergent with the POD modes. However, the interpolation error at $M = 0.8$ is 2~3 orders of magnitude higher than that at $M = 0.5$.



(a) $M = 0.8$



(b) $M = 0.5$

Fig. 7. Surface pressure coefficient of POD interpolation and CFD

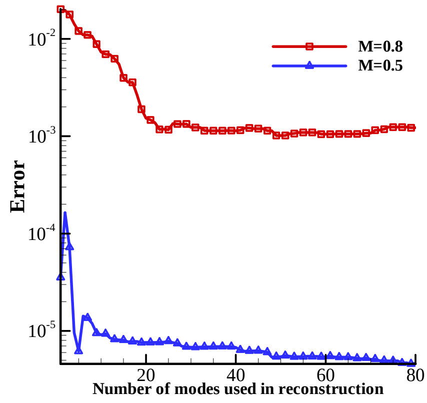
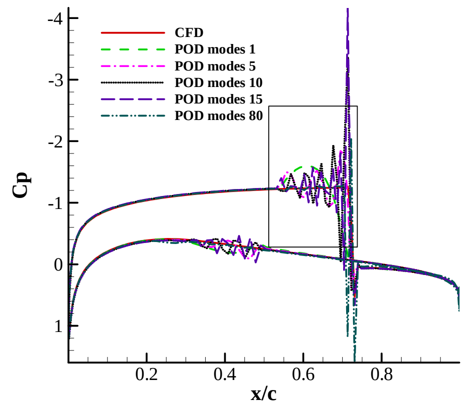


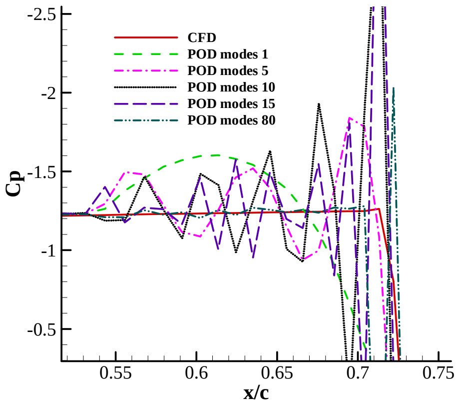
Fig. 8. Prediction error curve of POD interpolation

3.2.2 Extrapolation

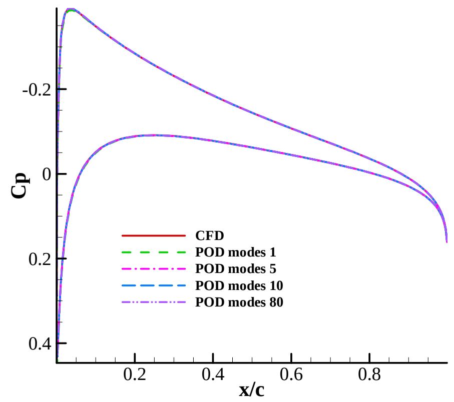
Figure 9 shows the comparison of surface pressure coefficient between POD extrapolation and CFD. It can be seen that oscillation phenomenon still occurs at $M = 0.8$. But if POD modes are increased, the oscillation amplitude cannot be decreased. It also can be seen from figure 10 that the prediction curve with POD modes is not convergent. When POD modes used in the prediction are increased, the prediction error increases instead of decrease. However, the extrapolation prediction error curve at $M = 0.5$ converges quickly and maintains at a low value. Therefore, the shock discontinuity in the flow destroys the prediction ability of POD method combined with extrapolation so that it no longer has the extrapolation ability.



(a) $M = 0.8$



(b) Close-up view of $M = 0.8$



(c) $M = 0.5$

Fig. 9. Surface pressure coefficient of POD extrapolation and CFD

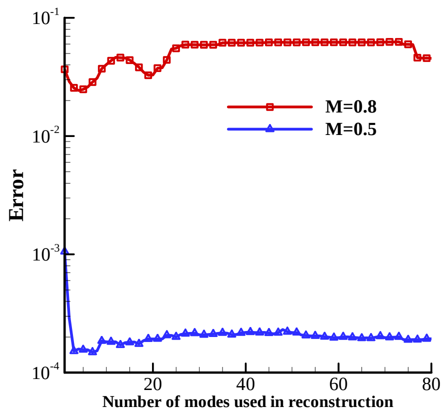


Fig. 10. Prediction error curve of POD extrapolation

4 Conclusion

For the flow with the shock discontinuity, POD method is used to get POD modes. And then the snapshot is projected to each POD mode to obtain modal coefficients, so as to realize the reconstruction of flow field. The cubic spline interpolation is conducted on modal coefficients for interpolation or extrapolation to predict flow solution which is not contained in the snapshots set. The reconstruction accuracy and prediction ability of the POD method are studied in the view of the discontinuous flow field and following conclusions can be drawn:

(1) In the subsonic flow, only one POD mode are needed to reconstruct the snapshots with high precision. While in the transonic flow, outside the shock wave region, one POD mode can result a good reconstruction. But if we want to reconstruct the shock wave region precisely, more POD modes are needed.

(2) POD method combined with interpolation or extrapolation is a good means to predict flow solutions for snapshots without the shock discontinuity. However, for snapshots with the shock discontinuity, although this method still has a relatively good interpolation prediction ability, it does not have an extrapolation prediction ability.

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