MATHEMATICAL MODELING OF A LOW-POWER GAS TURBINE ENGINE AND ITS CONTROL SYSTEM

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Abstract

In this paper a non-linear mathematical model of a low-power gas turbine, which is used in the Budapest University of Technology and Economics, will be presented.

This topic is current, because with this model the manoeuvrerability of the engine can increase in special flight situations.

According to the measurements, the parameters of the gas turbine are known, determined.

The method of non-linear modelling is based on the thermodynamical equations, which describe the behaviour of the engine.

With this model after linearization we can design an optimal controller for the gas turbine.

1 State-space representation

The signals are time-dependent functions, which can be scalar-valued or vector-valued, and deterministic or stochastic. In the following only vector-valued and deterministic signals will be used.

The system is any part of the real world surrounded by a well-defined boundary. The system is influenced by its environment via signals (u(t) - input signal), and acts on its environment by other signals (y(t) – output signal).

The state of the system contains all past information on the system up to time t_0 – time instant. If we would like to compute the output signal for $t \geq t_0$ (all future values) we only need u(t), $t \geq t_0$ and the state $t=t_0$.

A description, which uses the state signal, is called state space description, or state space representation.

The general form of the state space representation of a finite dimensional linear time invariant system is:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \text{- state equation} \qquad (1)$$
$$y(t) = Cx(t) + Du(t) \quad \text{- output equation,} \qquad (2)$$

with given initial condition $x(t_0)=x(0)$, and

$$x(t) \in R^n, \qquad y(t) \in R^p, \qquad u(t) \in R^r \qquad (3)$$

being vectors of finite dimensional spaces, and

$$A \in R^{n \times n}, \qquad B \in R^{n \times r},$$
$$C \in R^{p \times n}, \qquad D \in R^{p \times r}. \qquad (4)$$

The general form of the state space representation of a finite dimensional non-linear time invariant system is:

$$\dot{x}(t) = f(x(t), u(t)) \quad \text{- state equation} \qquad (5)$$
$$y(t) = h(x(t), u(t)) \quad \text{- output equation,} \qquad (6)$$

with the vector-values state, input and output vectors x, u and y.

2 Description of the gas turbine

The subject of our analysis is a low-power single-spool gas turbine with single-stage centrifugal compressor and single-stage centripetal turbine. This engine is a special design for technical colleges, universities. According to this attribute the main principles of construction to be observed were simplicity of design and ease of maintenance. Consequently, the unit was designed with small power-to-weight ratio, limited space requirement, multi-fuel capabilities and vibration-free running.

The most important parameters of the engine (if $p_0 = 1{,}0133$ bar, $T_0 = 288$K, n = 50000 1/min):

Power:	P = 80 kW,
Mass flow rate of air:	$m_c = 0{,}9$ kg/sec,
Compressor pressure ratio:	$\pi_c^* = 2{,}8$,
Exhaust gas temperature:	$T_4^* = 938$ K.

We are able to measure:

- total pressure and total temperature
  - before and after the compressor (p_1^*, p_2^* and T_1^*, T_2^*) ,
  - before and after the turbine (p_3^*, p_4^* and T_3^*, T_4^*),
- number of revolutions (n),
- consumption of fuel (m_{fuel}),
- and the moment of load (M_{load})

3 Thermodynamic non-linear mathematical model

The thermodynamic non-linear model describes the behaviour of the engine. It has two different parts: the first contains the steady-state equations and the second contains the dynamic equations.

The steady-state equations describe the working of the elements of the engine in an operating (equilibrium) point. Our gas turbine has four parts.

1. The first part is the inlet. In this component, it is assumed that the flow is adiabatic. Let the pressure loss is σ_{inlet}, the adiabatic exponent is κ, the Mach number is M and the parameters (pressure and temperature) p_0 and T_0! With them the conditions (total temperature and pressure) at the inlet to the compressor are:

$$T_1^* = T_0^* = T_0 \cdot (1 + \frac{\kappa - 1}{2} M^2) \tag{7}$$

$$p_1^* = \sigma_{inlet} \cdot p_0^* = \sigma_{inlet} \cdot p_0 \cdot (1 + \frac{\kappa - 1}{2} M^2)^{\frac{\kappa}{\kappa - 1}} \tag{8}$$

2. The second part is the compressor. It is modelled empirically using the compressor characteristic (map), which gives the mass flow across the compressor. The total temperature rise is found by using the isentropic efficiency factor η_c, the total pressure is the function of compressor pressure ratio π_c^* and n is the number of revolution:

$$\dot{m}_c = f(n, p_2^*, p_1^*, T_1^*) \tag{9}$$

$$T_2^* = T_1^* \cdot \left(1 + \frac{\left(\frac{p_2^*}{p_1^*} \right)^{\frac{\kappa - 1}{\kappa}} - 1}{\eta_c} \right) \tag{10}$$

$$p_2^* = \Pi_c \cdot p_1^* \tag{11}$$

3. The third part is the combustor. In the combustor the total pressure loss (σ_f) is assumed to be a fixed percentage of its inlet pressure p_2^* and total temperature rise is given by the steady-state energy equation, in which c_p is the heat capacity of air in constant pressure, η_{comb} is the efficiency factor of combustion and H_l is the lower thermal value of fuel, in the combustor:

$$p_3^* = \sigma_f \cdot p_2^* \tag{12}$$

$$T_3^* = \frac{T_2^* + q_T \cdot \frac{Q_f \cdot \eta_f}{c_p}}{1 + q_T} \tag{13}$$

$$\text{where: } q_T = \frac{\dot{m}_{fuel}}{\dot{m}_c} \tag{14}$$

4. The last part is the turbine. It is modelled empirically with the steady-state turbine performance map, which gives the mass

flow across the turbine. The total temperature drop is found by using the isentropic efficiency factor η_t, and total pressure is the function of pressure loss σ_g in the tube after the turbine:

$$\dot{m}_t = \frac{\beta \cdot A_3 \cdot q(\lambda_3) \cdot p_3^*}{\sqrt{T_3^*}} \tag{15}$$

$$T_4^* = T_3^* \cdot \left(1 - \eta_t \cdot \left(1 - \frac{1}{\left(\frac{p_3^*}{p_4^*} \right)^{\frac{\kappa-1}{\kappa}}} \right) \right) \tag{16}$$

$$p_4^* = \sigma_g \cdot p_0 \tag{17}$$

The dynamic equations describe the changes in the gas turbine, when the operating point is varying because of the operator or disturbances.

1. The first dynamic equation comes from the power balance on the compressor/turbine spool, where P_l is from the loading:

$$4 \cdot \Pi^2 \cdot \Theta \cdot n \cdot \frac{dn}{dt} = P_t \cdot \eta_m - P_c - P_l = \\ = \dot{m}_t \cdot c_p \cdot \left(T_3^* - T_4^*\right) \cdot \eta_m - m_c \cdot c_p \cdot \left(T_2^* - T_1^*\right) - P_l \tag{18}$$

2. A control volume is needed around the combustor to model the dynamic behaviour. For this control volume a mass balance and an energy balance produced two first-order differential equations:

- mass balance

$$\frac{dp_2^*}{dt} = \frac{R \cdot T_2^*}{V_2} \cdot \left(\dot{m}_c + \dot{m}_{fuel} - \dot{m}_t\right) \tag{19}$$

- energy balance

$$\frac{dT_3^*}{dt} = \frac{1}{m_{comb}} \cdot \\ \cdot \left(\left(\dot{m}_c + \dot{m}_{fuel}\right) \cdot T_2^* - \dot{m}_t \cdot T_3^* \right) \cdot \kappa - \left(\dot{m}_c + \dot{m}_{fuel} - \dot{m}_t\right) \cdot T_3^* \tag{20}$$

4 Jacobian linearization

It was shown, that a gas turbine can be represented by the following non-linear differential and algebraic equations:

$$\dot{x} = F(x,u) \\ y = G(x,u) \tag{21}$$

In deriving linear models, we assume that functions F and G are continuous and differentiable. If the system described by Eq. (21) is in a steady-state condition, when constant input u_{ss} producing constant state x_{ss}, and constant output y_{ss}, then the combination (u_{ss}, x_{ss}, y_{ss}) satisfies:

$$0 = F(x_{ss}, u_{ss}) \\ y_{ss} = G(x_{ss}, u_{ss}) \tag{22}$$

The point (u_{ss}, x_{ss}, y_{ss}) is an equilibrium point of the gas turbine. Perturbating the control input with δu results in state and output perturbation δx and δy, respectively and control input, state and output become $u = u_{ss} + \delta u$, $x = x_{ss} + \delta x$ and $y = y_{ss} + \delta y$, and the Eq. (22) follows:

$$\dot{x}_{ss} + \delta\dot{x} = F(x_{ss} + \delta x, u_{ss} + \delta u) \\ y_{ss} + \delta y = G(x_{ss} + \delta x, u_{ss} + \delta u) \tag{23}$$

Because of the continuity requirements imposed on the F and G functions Eq (23) can be expanded in Taylor series about the point (u_{ss}, x_{ss}, y_{ss}). Ignoring the higher order items, the results is:

$$\delta\dot{x} = A \cdot \delta x + B \cdot \delta u \\ \delta y = C \cdot \delta x + D \cdot \delta u \tag{24}$$

The constant matrices A, B, C and D have the dimensions of $n \times n$, $n \times m$, $r \times n$ and $r \times m$, and given by:

$$A_{ij} = \frac{\delta F_i}{x_j}, \quad B_{ij} = \frac{\delta F_i}{u_j}, \quad C_{ij} = \frac{\delta G_i}{x_j}, \quad D_{ij} = \frac{\delta G_i}{u_j} \tag{25}$$

This equation approximates the dynamic behaviour of the non-linear gas turbine in a small region about the operating point.

5 State space representation of our gas turbine

In our case the state, input and output vectors will be the follows:

$$\underline{u} = \begin{bmatrix} \dot{m}_{fuel} \\ P_l \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} n \\ p_2^* \\ T_3^* \end{bmatrix}, \quad \underline{y} = \begin{bmatrix} n \\ p_2^* \\ T_3^* \end{bmatrix}. \tag{26}$$

To construct the A, B, C, D matrices we have to derive the dynamic equations. For example:

$$\frac{dp_2^*}{dt} = \frac{R \cdot T_2^*}{V_2} \cdot \left(\dot{m}_c(n, p_2^*) + \dot{m}_{fuel} - \frac{\beta \cdot A_3 \cdot q(\lambda_3) \cdot p_3^*}{\sqrt{T_3^*}} \right) = F(n, p_2^*, T_3^*, \dot{m}_{fuel}, P_l) \tag{27}$$

The A_{21} parameter (the first element of the second row):

$$\frac{\partial F}{\partial n} = \frac{R \cdot T_2^*}{V_2} \cdot \frac{\partial \dot{m}_c}{\partial n}(p_2^* = const.) \tag{28}$$

The A_{22} parameter (the second element of the second row):

$$\frac{\partial F}{\partial p_2^*} = \frac{\kappa - 1}{\kappa} \cdot \frac{R \cdot T_0}{V_2} \cdot \frac{p_2^{*\frac{-1}{\kappa}}}{p_1^{*\frac{\kappa-1}{\kappa}}} \cdot \left(\dot{m}_c(n, p_2^*) + \dot{m}_{fuel} - \frac{\beta \cdot A_3 \cdot q(\lambda_3) \cdot p_3^*}{\sqrt{T_3^*}} \right) + \frac{R \cdot T_2^*}{V_2} \cdot \frac{\partial \dot{m}_c}{\partial p_2^*}(n = const.) - \frac{R \cdot T_2^*}{V_2} \cdot \frac{\beta \cdot A_3 \cdot q(\lambda_3) \cdot p_3^*}{\sqrt{T_3^*}} \tag{29}$$

The A_{23} parameter (the third element of the second row):

$$\frac{\partial F}{\partial T_3^*} = 0{,}5 \cdot \frac{R \cdot T_2^*}{V_2} \cdot \frac{\beta \cdot A_3 \cdot q(\lambda_3) \cdot p_3^*}{T_3^{*1,5}} \tag{30}$$

The B_{21} parameter (the first element of the second row):

$$\frac{\partial F}{\partial \dot{m}_{fuel}} = \frac{R \cdot T_2^*}{V_2} \tag{31}$$

The B_{22} parameter (the second element of the second row):

$$\frac{\partial F}{\partial P_l} = 0 \tag{32}$$

6 LQ control design

Given a linearised model corresponding to an engine operating point, the standard linear optimal design technique can be used to determine the full state feedback gains by minimising the <u>quadratic performance criteria</u>:

$$J = \frac{1}{2} \cdot \int_0^\infty (\delta x^T Q \delta x + \delta u^T R \delta u) dt, \tag{33}$$

and solving the corresponding Control Algebraic Riccati Equation (CARE):

$$PA + A^T P - PBR^{-1}B^T P + Q = 0, \tag{34}$$

the state feedback gains are:

$$\delta u = -R^{-1} B^{T} P \delta x = -K \delta x \tag{35}$$

This is the optimal control for the gas turbine.

7 Simulation

With the help of the equations above we measured and calculated the parameters, the elements of the A, B, C and D matrices. With these we can simulate the behaviour of the gas turbine, and we can design a controller for the engine.

In the next four diagrams the response-functions can be seen, when the input (or control input) is a Dirac-delta function. In the first two figures the system is without control, these are system-simulations. The next two is with the optimal LQ control.

8 Summary

Comparison:
The LQ control has changed the quantitative features of the gas turbine, the time constants are smaller, so we need less time for the system goes back to the operating point.

Possibility to continue this investigation:

This LQ control design is for an ideal system without any noises, disturbances. So it is important, that in the future the noises, disturbances and uncertainties of the system (because of the measuring and neglected dynamics) will have to be investigated, and with them we will be able to design a new controller.

References

[1] I. Sánta: Analysis of Non-Linear Thermal Models for Jet Engines (1988)

[2] O. F. Qi, N. R. L. Maccallum and P. J. Gawthrop: Improving Dynamic Response of a Single-Spool Gas Turbine Engine Using a Non-Linear Controller (1992)

[3] M. Athans: Linear-Quadratic Gaussian with Loop Transfer Recovery Methodology for the F-100 Engine (1986)

[4] J. W. Watts: Optimal State Space Control of a Gas Turbine Engine (1991)

[5] J. W. Watts: A Model and State-Space Controllers for an Intercooled, Regenerated (ICR) Gas Turbine Engine (1992)

[6] A. Biran, M. Breiner: MATLAB 5 for Engineers (1999)

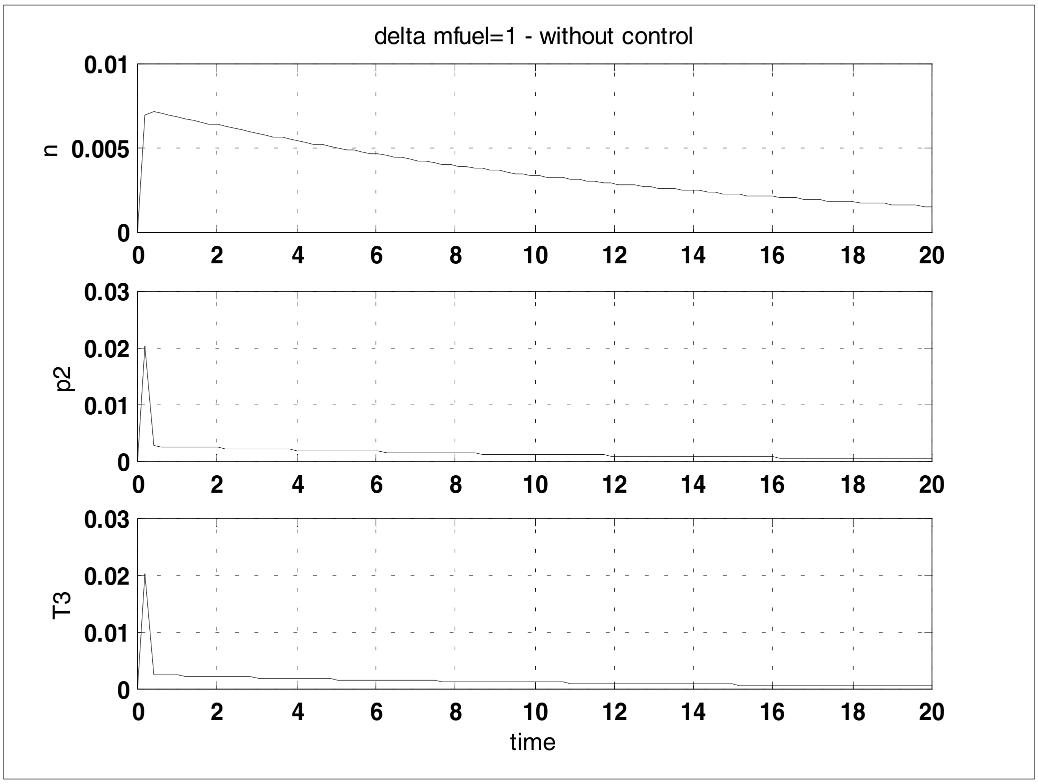

Fig. 1.: Simulation when m_{fuel} is a Dirac-delta function (without control)

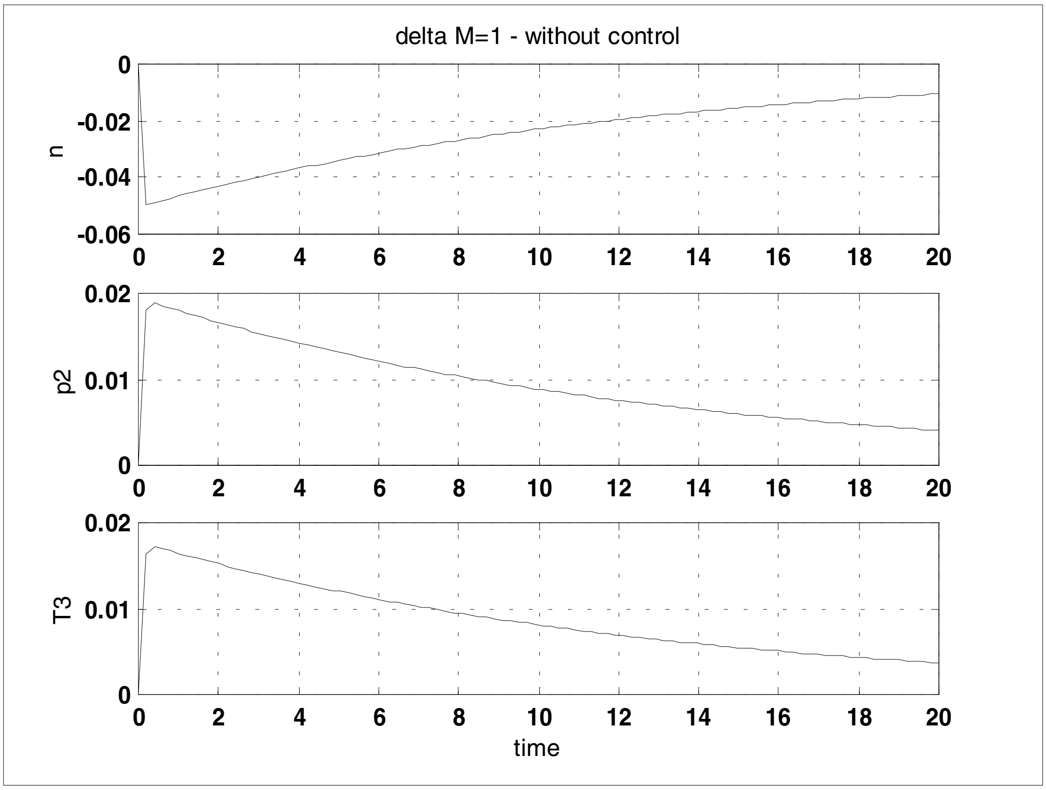

Fig. 2.: Simulation when P_l is a Dirac-delta function (without control)

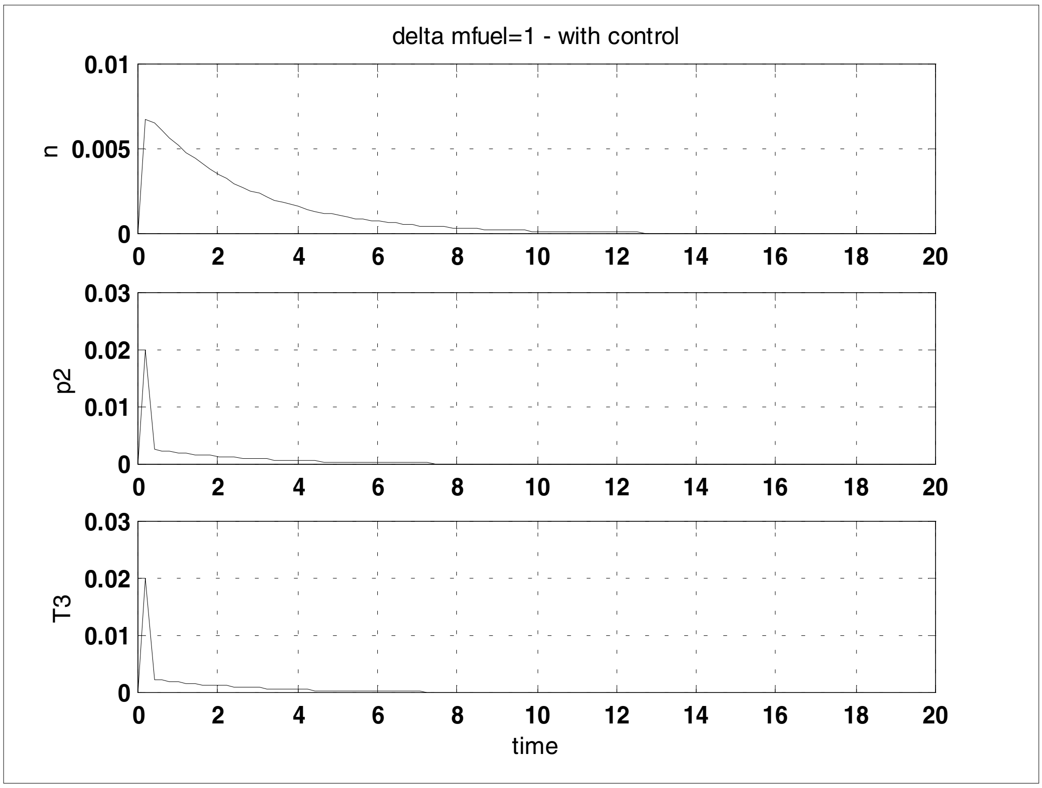

Fig.3.: Simulation when m_{fuel} is a Dirac-delta function (with LQ control)

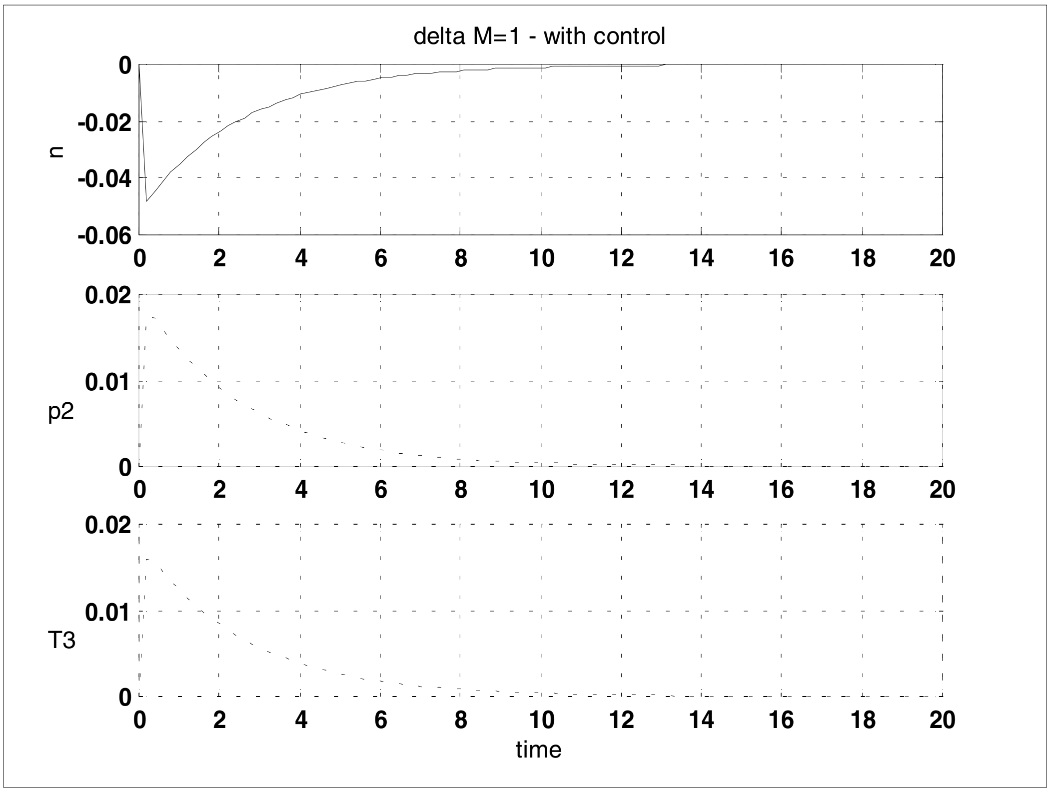

Fig. 4.: Simulation when P_l is a Dirac-delta function (with LQ control)