FLUTTER ANALYSIS OF COMPOSITE WINGS USING SYMBOLIC COMPUTATION

J.R. Banerjee* and R. Butler**

***Department of Mechanical Engineering and Aeronautics,**
City University, Northampton Square,
London EC1V 0HB, England, U.K., E-mail : J.R.Banerjee@city.ac.uk
****Department of Mechanical Engineering, University of Bath, Claverton Down,**
Bath BA2 7AY, England, U.K., E-mail : R.Butler@bath.ac.uk

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Abstract

Symbolic computation is used to investigate the flutter behaviour of uniform composite wings analytically instead of numerically. As a result the proposed method requires minimum computational effort because all numerical matrix manipulations associated with the solution of flutter problems are completely avoided. The wing is idealised as a bending-torsion (materially) coupled composite beam with cantilever end condition for which the frequency equation and mode shapes in free natural vibration are presented in closed analytical form. For a given number of selected normal modes, the expressions for generalised mass, generalised stiffness and generalised aerodynamic force are derived in explicit analytical form. This was assisted greatly by symbolic computation. Finally the flutter problem is formulated by summing algebraically the expressions for generalised mass, generalised stiffness and generalised aerodynamic force terms. From the final expression containing all the above terms the flutter speed and flutter frequency are determined by using a standard root finding procedure. As a consequence, the proposed analytical method is found to be accurate and efficient, and therefore, it holds out the prospect of precise aeroelastic optimisation. An illustrative example confirming the correctness and accuracy of the method when predicting the flutter speed and flutter frequency of a laminated composite wing is provided.

1.0 Introduction

With the advent of advanced materials such as fibrous composites, aeroelasticity of composite wings has become a major research topic in recent years. There are two important reasons that stimulate this research. First, composite materials have very high strength to weight ratios when compared with their metallic counterpart. Secondly, and more importantly from an aeroelastic point of view, they have directional properties which can be manipulated to achieve desirable aeroelastic characteristics. A number of publications [1-5] illustrate the flutter behaviour of laminated composite wings by using normal modes obtained from beam element idealisation of the wing and strip theory based on Theodorsen type unsteady aerodynamics [6]. It has been found that significant parameters that affect the flutter speed are the laminate stacking sequence, ply orientation and sweep angle. Investigations on the subject have shown significant benefits that can not be achieved easily for wings with metallic construction. For instance, swept forward wing design with acceptable divergence speed is now a distinct possibility. Because of these potential possibilities, research in the area of aeroelastic optimisation, often called aeroelastic tailoring [7-9] has flourished, and is still continuing. An optimisation study is computationally intensive by its nature, and any improvement in the method of flutter analysis or solution technique that can enhance either whole or part of the procedures (for example sensitivity calculations) is naturally welcome by research workers. In this paper an analytical method of flutter analysis for a uniform cantilever composite wing is developed for the first time. Such a development has been possible due to recent advancements in symbolic computing [10].

Banerjee [11] has recently developed exact analytical expressions for the frequency equation and mode shapes of a uniform composite beam with cantilever end condition by making extensive use of the symbolic algebraic package REDUCE [12]. The formulae for frequency equation and mode shapes derived by Banerjee [11] account for the bending-torsion material coupling effect that is usually prevalent in composite beams due to ply orientations. The work of Banerjee [11] is extended further in this paper to cover flutter analysis by linking the modal analysis to unsteady aerodynamic analysis of the wing forces. First, the analytically derived normal modes are implemented in the expressions for the generalised mass and generalised stiffness in a particular mode. The symbolic algebraic (manipulative) package REDUCE [12] is then used to obtain the integral expressions associated with the generalised mass and generalised stiffness terms in explicit form. The derivation of the generalised aerodynamic forces in explicit analytical form in terms of normal co-ordinates is a difficult task. The difficulty arises because the unsteady aerodynamic forces consist of expressions that have both real and imaginary parts [6]. Due to advancements in symbolic computation in recent years [10], this became possible and the work has thus been greatly assisted by REDUCE [12]. Once the analytical expressions for generalised mass, generalised stiffness and generalised aerodynamic force in each mode are obtained individually in explicit form, they are summed algebraically to formulate the complex flutter function which is primarily a function of two unknown variables, namely the air-speed and the frequency. The zeros of this function, which give the flutter speed and flutter frequency, are obtained by a standard root finding procedure for the real and imaginary parts of the function. Only in this final stage does the problem become numerical in the sense that results are obtained from the roots of an analytical function rather than from more conventional numerical matrix manipulation. The above theory has been applied to predict the flutter speed of a composite cantilever wing for which some comparative results are available in the literature. The expected accuracy of the proposed theory is confirmed by numerical results.

2.0 THEORY

Composite wings have been characterised in the literature [1-5] by their rigidities which are essentially the bending rigidity EI, torsional rigidity GJ, and bending-torsion (material) coupling rigidity K. The material coupling rigidity K which does not exist in metallic wings, is of great significance for composite wings because it can be exploited to advantage for aeroelastic tailoring purposes.

A uniform composite wing of length L, bending rigidity EI, torsional rigidity GJ, bending-torsion (material) coupling rigidity K, mass per unit length m, and mass moment of inertia per unit length I_α, is shown in Fig. 1. In the right handed co-ordinate system shown, the centroidal axis which is coincident with the Y-axis, is allowed to deflect out of the plane by h(y,t), whilst the cross-section is allowed to rotate (or twist) about OY by ψ(y,t), where y and t denote distance from the origin and time respectively. The wing, with cantilever boundary conditions and with the built-in end chosen as the origin, is assumed to undergo simple harmonic oscillation with circular (or angular) frequency ω.

2.1 Free vibration analysis

2.1.1 Natural frequencies

In accordance with the recent work of Banerjee [11], the frequency equation for the composite wing shown in Fig. 1 with cantilever end conditions is given by Equation (1) below, provided that appropriate substitutions are made from Equations (2)-(16), and that each of the parameters is evaluated in the given order of sequence by making use of the beam properties EI, GJ, K, m, I_α , L, and a chosen value of the trial frequency ω. It should be noted that the left hand side of the frequency equation given by Equation (1) is primarily a function of the frequency ω, the zeros of which gives the natural frequencies ω_n in free vibration.

$$f(\omega) = \lambda_1 C_\beta C_\gamma C_{h\alpha} + \lambda_2 C_\beta S_\gamma S_{h\alpha} + \lambda_3 C_\gamma S_\beta S_{h\alpha} + \lambda_4 S_\beta S_\gamma C_{h\alpha} + \xi_1 C_\beta + \xi_2 C_\gamma + \xi_3 C_{h\alpha} = 0 \tag{1}$$

where

$$\bar{a} = I_\alpha \omega^2 L^2/GJ,\ \bar{b} = m\omega^2 L^4/EI;\ \bar{k} = K/EI \tag{2}$$

$$c = 1 - K^2/(EI\ GJ) \tag{3}$$

$$a = \bar{a}/c,\ b = \bar{b}/c \tag{4}$$

$$q = b + a^2/3 \tag{5}$$

$$\phi = \cos^{-1}[(27abc - 9ab - 2a^3)/\{2(a^2+3b)^{3/2}\}] \tag{6}$$

$$\alpha = [2(q/3)^{1/2}\cos(\phi/3) - a/3]^{1/2},$$
$$\beta = [2(q/3)^{1/2}\cos\{(\pi-\phi)/3)\} + a/3]^{1/2},$$
$$\gamma = [2(q/3)^{1/2}\cos\{(\pi+\phi)/3)\} + a/3]^{1/2} \tag{7}$$

$$\bar{\alpha} = \bar{b}/\alpha^2,\ \bar{\beta} = \bar{b}/\beta^2,\ \bar{\gamma} = \bar{b}/\gamma^2 \tag{8}$$

$$k_\alpha = (\bar{b} - \alpha^4)/\bar{k}\alpha^3,\ k_\beta = (\bar{b} - \beta^4)/\bar{k}\beta^3,$$
$$k_\gamma = (\bar{b} - \gamma^4)/\bar{k}\gamma^3 \tag{9}$$

$$g_\alpha = (\bar{b} - c\alpha^4)/\bar{k}\alpha^2,\ g_\beta = (\bar{b} - c\beta^4)/\bar{k}\beta^2,$$
$$g_\gamma = (\bar{b} - c\gamma^4)/\bar{k}\gamma^2 \tag{10}$$

$$C_{h\alpha} = \cosh\alpha;\ C_\beta = \cos\beta;\ C_\gamma = \cos\gamma \tag{11}$$

$$S_{h\alpha} = \sinh\alpha;\ S_\beta = \sin\beta;\ S_\gamma = \sin\gamma \tag{12}$$

$$\mu_1 = \alpha k_\beta - \beta k_\alpha,\ \mu_2 = \beta k_\gamma - \gamma k_\beta,$$
$$\mu_3 = \gamma k_\alpha - \alpha k_\gamma \tag{13}$$

$$\nu_1 = \bar{\alpha} g_\beta - \bar{\beta} g_\alpha,\ \nu_2 = \bar{\beta} g_\gamma - \bar{\gamma} g_\beta,$$
$$\nu_3 = \bar{\gamma} g_\alpha - \bar{\alpha} g_\gamma \tag{14}$$

$$\lambda_1 = -2\alpha\bar{\alpha}\mu_2\nu_2 - \alpha\bar{\gamma}\mu_2\nu_1 - \bar{\alpha}\gamma\mu_1\nu_2 - 2\gamma\bar{\gamma}\mu_1\nu_1,$$
$$\lambda_2 = -\alpha\bar{\alpha}\mu_1\nu_2 + \gamma\bar{\gamma}\mu_2\nu_1,$$
$$\lambda_3 = -\alpha\bar{\alpha}\mu_3\nu_2 - \bar{\alpha}\beta\mu_2\nu_2 - \beta\bar{\gamma}\mu_2\nu_1,$$
$$\lambda_4 = -\bar{\alpha}\beta\mu_1\nu_2 - \beta\bar{\gamma}\mu_1\nu_1 + \gamma\bar{\gamma}\mu_3\nu_1 \tag{15}$$

$$\xi_1 = \alpha\bar{\alpha}\mu_2\nu_1 + \gamma\bar{\gamma}\mu_1\nu_2,$$
$$\xi_2 = \alpha\bar{\alpha}\mu_2\nu_3 - \alpha\bar{\beta}\mu_2\nu_2 - \bar{\beta}\gamma\mu_1\nu_2,$$
$$\xi_3 = \alpha\bar{\beta}\mu_2\nu_1 + \bar{\beta}\gamma\mu_1\nu_1 - \gamma\bar{\gamma}\mu_1\nu_3 \tag{16}$$

It can be verified [11] that the value of f(ω) in Equation (1) is zero when ω=0, which corresponds to a composite wing at rest so that there is no inertial loading on the wing. This known value of f(0)=0 can be used to avoid any numerical problem of overflow at zero frequency when computing f(ω). For all other (non-trivial) values of ω, the expression for f(ω) given by Equation (1) can be used when locating the natural frequencies by tracking successively the changes of its sign.

2.1.2 Mode shapes

Once the natural frequencies ω_n have been found from Equation (1), the normal mode shapes for the wing consisting of bending displacement (H_n) and torsional rotation (Ψ_n) can be expressed as [11]

$$H_n(\xi) = A_n\cosh\alpha_n\xi + B_n\sinh\alpha_n\xi + C_n\cos\beta_n\xi + D_n\sin\beta_n\xi + E_n\cos\gamma_n\xi + F_n\sin\gamma_n\xi \tag{17}$$

and

$$\Psi_n(\xi) = P_n\cosh\alpha_n\xi + Q_n\sinh\alpha_n\xi + R_n\cos\beta_n\xi + S_n\sin\beta_n\xi + T_n\cos\gamma_n\xi + U_n\sin\gamma_n\xi \tag{18}$$

where $\xi = y/L$ is the non-dimensional spanwise distance from the wing root, and α_n , β_n and γ_n are calculated from Equations (7) with the help of Equations (2)-(6) by substituting ω_n in place of ω.

It has been shown earlier by Banerjee [11] that the coefficients A_n, B_n, C_n, D_n, E_n, F_n are related to P_n, Q_n, R_n, S_n, T_n, U_n as follows

$$P_n = (k_\alpha/L)B_n;\ Q_n = (k_\alpha/L)A_n;\ R_n = (k_\beta/L)D_n;$$
$$S_n = -(k_\beta/L)C_n;\ T_n = (k_\gamma/L)F_n;\ U_n = -(k_\gamma/L)E_n \tag{19}$$

where k_α, k_β and k_γ have already been defined in Equations (9) but must be calculated using α_n , β_n

and γ_n.

Also for cantilever end conditions of the wing, the ratios of the mode shape coefficients in terms of A_n are given by (see Ref. 11)

$$B_n/A_n=(-\beta\ \bar{\beta}\mu_2\zeta_3S_\beta+\gamma\ \bar{\gamma}\mu_2\zeta_1S_\gamma-\alpha\ \bar{\alpha}\mu_2\zeta_2S_{h\alpha})/\chi,$$

$$C_n/A_n=(\ \bar{\alpha}\ \bar{\beta}\delta_3C_{h\alpha}C_\beta+\ \bar{\alpha}\ \bar{\gamma}\tau_2C_{h\alpha}C_\gamma-\ \bar{\beta}\mu_3\varepsilon_3S_\beta+$$

$$\bar{\alpha}\ \bar{\gamma}\tau_1S_{h\alpha}S_\gamma+\ \bar{\beta}\ \bar{\gamma}\delta_3C_\beta C_\gamma+\tau_3)/\chi$$

$$D_n/A_n=(\ \bar{\alpha}\ \bar{\beta}\delta_3C_{h\alpha}S_\beta+\gamma\ \bar{\gamma}\mu_3\zeta_1S_\gamma-\alpha\ \bar{\alpha}\mu_3\zeta_2S_{h\alpha}+$$

$$\bar{\beta}\ \bar{\gamma}\delta_3S_\beta C_\gamma)/\chi$$

$$E_n/A_n=\{-\ \bar{\alpha}\ \bar{\beta}(\delta_3+\alpha\mu_2)C_{h\alpha}C_\beta+\gamma\ \bar{\gamma}\mu_1\zeta_1C_\gamma-$$

$$\bar{\alpha}\ \bar{\beta}\delta_1S_{h\alpha}S_\beta-\ \bar{\gamma}\mu_1\varepsilon_1S_\gamma-\alpha\ \bar{\alpha}^2\mu_2-\ \bar{\beta}^2\delta_3\}/\chi$$

$$F_n/A_n=(-\beta\ \bar{\beta}\mu_1\zeta_3S_\beta+\gamma\ \bar{\gamma}\mu_1\zeta_1S_\gamma-\alpha\ \bar{\alpha}\mu_1\zeta_2S_{h\alpha})/\chi \qquad (20)$$

where μ_1, μ_2 and μ_3 have already defined in Equations (13), and the following additional variables are introduced.

$$\zeta_1=\ \bar{\alpha}_nC_{h\alpha}+\ \bar{\beta}_nC_\beta,\ \zeta_2=\ \bar{\beta}_nC_\beta-\ \bar{\gamma}_nC_\gamma,$$

$$\zeta_3=\ \bar{\gamma}_nC_\gamma+\ \bar{\alpha}_nC_{h\alpha} \qquad (21)$$

$$\varepsilon_1=\alpha_n\ \bar{\alpha}_nS_{h\alpha}-\beta_n\ \bar{\beta}_nS_\beta,\ \varepsilon_2=\beta_n\ \bar{\beta}_nS_\beta-\gamma_n\ \bar{\gamma}_nS_\gamma,$$

$$\varepsilon_3=\gamma_n\ \bar{\gamma}_nS_\gamma-\alpha_n\ \bar{\alpha}_nS_{h\alpha} \qquad (22)$$

$$\delta_1=\alpha_n\mu_3+\beta_n\mu_2,\ \delta_2=\beta_n\mu_1-\gamma_n\mu_3,\ \delta_3=\gamma_n\mu_1+\alpha_n\mu_2 \qquad (23)$$

$$\tau_1=\alpha_n\mu_1+\gamma_n\mu_2,\ \ \tau_2=\alpha_n\mu_2-\gamma_n\mu_1$$

$$\tau_3=\alpha_n\ \bar{\alpha}_n^{\ 2}\mu_2-\gamma_n\ \bar{\gamma}_n^{\ 2}\mu_1 \qquad (24)$$

$$\chi=\alpha_n\ \bar{\alpha}_n\mu_2\zeta_2C_{h\alpha}+\ \bar{\alpha}_n\mu_2\varepsilon_2S_{h\alpha}-\ \bar{\beta}\ \bar{\gamma}(\delta_3+\gamma\mu_1)C_\beta C_\gamma-$$

$$\bar{\beta}\ \bar{\gamma}\delta_2S_\beta S_\gamma+\ \bar{\beta}^2\delta_3+\gamma\ \bar{\gamma}^{\,2}\mu_1 \qquad (25)$$

Note that $C_{h\alpha}$, $S_{h\alpha}$, C_β, S_β, C_γ, S_γ and all other parameters appearing in Equations (19)-(25) above must be calculated at the natural frequency ω_n.

The coefficients P_n, Q_n, R_n, S_n, T_n, U_n which give the torsional displacements in the n-th mode (see Equation (18)) may also be expressed in terms of A_n using Equations (19) and (20). Thus the mode shape is completely defined in terms of A_n, which can be arbitrarily chosen (e.g. A_n=1).

2.2 Generalised Mass and Generalised Stiffness

The generalised mass M_n and generalised stiffness K_n in the n-th mode of the cantilever wing can be derived using the procedure put forward by Bishop and Price [13]. These are respectively given by

$$M_n=\int_0^1 (mH_n^2+I_\alpha\Psi_n^2)d\xi \qquad (26)$$

and

$$K_n=\int_0^1 [(EI(H_n^{//})^2+GJ(\Psi^{/})^2]d\xi \qquad (27)$$

where H_n and Ψ_n are given by Equations (17) and (18) and the terms EI, GJ, m, and I_α have been defined before.

However, a simpler means of calculating the generalised stiffness, K_n, would be to use the following equation

$$K_n=\omega_n^2M_n \qquad (28)$$

where ω_n, the n-th natural frequency has already been calculated from the frequency equation (see Equation (1)) prior to the calculation of mode shapes H_n and Ψ_n.

Using Equations (16) and (17), the integrals $\int_0^1 H_n^2\ d\xi$, $\int_0^1 \Psi_n^2\ d\xi$, $\int_0^1 H_n\Psi_nd\xi$ and $\int_0^1 H_n\Psi_md\xi$ are evaluated in explicit analytical form by performing symbolic computation with REDUCE [12]. The explicit expressions for these integrals in their most general forms are given in the Appendix.

2.3 Generalised Aerodynamic Coefficients

The generalised aerodynamic coefficients are derived by application of the principle of virtual work. The aerodynamic strip theory based on Theodorsen expressions for unsteady lift and pitching moment [6] and the normal modes obtained from the analytical formulae of free vibration theory explained above, are used when applying the principle of virtual work. Thus if the bending displacement and torsional rotation in the i-th mode are $H_i(\xi)$ and $\Psi_i(\xi)$ and U, b, ρ, k, C(k) and a_h are in the usual notation: the airspeed, semi-chord, density of air, reduced frequency parameter (defined as $k=\omega b/U$), Theodorsen function and elastic axis location from mid-chord respectively [6], the elements of the generalised aerodynamic matrix [QA] are given by

$$QA_{ij}=\int_0^1 (A_{11}H_iH_j+A_{12}H_j\Psi_i$$

$$+A_{21}H_i\Psi_j+A_{22}\Psi_i\Psi_j)d\xi \qquad (29)$$

where

$$A_{11}=-\pi\rho U^2\{-k^2+2C(k)ik\}$$

$$A_{12}=\pi\rho U^2b[(a_hk^2+ik)+2C(k)\{1+ik(0.5-a_h)\}]$$

$$A_{21}=-\pi\rho U^2b\{2C(k)ik(0.5+a_h)-k^2a_h\}$$

and

$$A_{22}=\pi\rho U^2b^2[2(0.5+a_h)C(k)\{1+ik(0.5-a_h)\}+$$

$$k^2/8 + k^2a_h^2+(a_h-0.5)ik] \qquad (30)$$

Note that the signs of A_{11} and A_{21} have been reversed because, unlike the sign convention used in Ref. 6, H is considered to be positive upward in this paper.

The elements of the generalised aerodynamic matrix [QA] are complex with each element having a real part and an imaginary part. This is as a consequence of the terms A_{11}, A_{12}, …etc in Equation (29) being complex (see Equations (30)). By contrast, the generalised mass and stiffness terms (see Equations (26)-(28)) are both real. Analytical expressions for each of the integrals in Equation (29) are obtained by use of REDUCE [12] (see Appendix for the most general cases of these integrals). The real and imaginary parts of the complex terms A_{11}, A_{12}, A_{21} and A_{22} in Equation (29) are dependent on the Theodorsen function C(k), see Equations (30), which can be expressed in the following form

$$C(k) = F + iG \qquad (31)$$

where F and G are real functions of the variable k given by [6]

$$F=\{J_1(J_1+Y_0)+Y_1(Y_1-J_0)\}/\{(J_1+Y_0)^2+(Y_1-J_0)^2\}$$

$$G=-(Y_1Y_0+J_1J_0)/\{(J_1+Y_0)^2+(Y_1-J_0)^2\} \qquad (32)$$

while J_0 , J_1 , Y_0 , Y_1 are standard Bessel functions of first and second kinds, of argument k.

Thus with the help of Equation (31) the real and imaginary parts of the terms A_{11}, A_{12}, A_{21} and A_{22} in Equations (30) can be expressed as

$$A_{11R}=\pi\rho U^2(k^2+2kG)\ ; \ A_{11I}=-2\pi\rho U^2kF$$

$$A_{12R}=\pi\rho U^2b\{a_hk^2+2F-2kG(0.5-a_h)\},$$

$$A_{12I}=\pi\rho U^2b\{k+2G+2kF(0.5-a_h)\},$$

$$A_{21R}=\pi\rho U^2b\{kG(1+2a_h)+k^2a_h\}\ ,$$

$$A_{21I}=-\pi\rho U^2bkF(1+2a_h),$$

$$A_{22R}=\pi\rho U^2b^2[2(0.5+a_h)\{F-kG(0.5-a_h)\}+$$

$$k^2/8+k^2a_h^2],$$

and

$$A_{22I}=\pi\rho U^2b^2[2(0.5+a_h)\{G+kF(0.5-a_h\}-$$

$$k(0.5-a_h)] \qquad (33)$$

where the suffices R and I stand for the real and imaginary parts of the coefficients respectively.

2.4 Formulation of the Flutter Problem

Using the standard classical approach, the flutter determinant is formed from the flutter matrix, and this is achieved by summing algebraically the generalised mass, generalised stiffness and the generalised aerodynamic matrices. Thus for a system without structural damping the flutter matrix [QF] for n number of modes can be expressed in the form given by Equation (34) below. (Structural damping has most often a small effect on the oscillatory motion and is not included here.)

$$[QF]\{q\}=[\,-\omega^2[M]+[K]-[QA]\,]\{q\} \qquad (34)$$

where [QA] is the complex n×n generalised aerodynamic matrix defined in Equation (29), [M] and [K] are n×n diagonal matrices of generalised mass and generalised stiffness respectively (with the i-th diagonal representing the generalised mass M_i and generalised stiffness K_i), {q} is the column vector of n generalised co-ordinates and ω is the circular frequency in rad/s.

For flutter to occur, the determinant of the complex flutter matrix must be zero so that from Equation (34)

$$|QF| = |\,-\omega^2[M]+[K]-[QA]\,| = 0$$

The solution of the flutter determinant can now be sought by expanding the above determinant in algebraic form because each of the terms of [M], [K] and [QA], and hence each of the elements of [QF], are now available in analytical form.

3.0 RESULTS AND DISCUSSION

An illustrative example is chosen which examines the cantilever carbon-epoxy composite wing of Ref. 3. The wing is assumed to be unswept and is modelled by using a total number of 14 plies which are all orientated along a common direction θ, so that the stacking sequence is $[\theta]_{14}$. The first three natural frequencies and mode shapes were established by using the explicit frequency equation (1) and the mode shape expressions given by equations (17) and (18). These are shown in Fig.2. The first mode shows a strong coupling between the bending displacement and torsional rotation. The second mode is dominated by torsional rotation with a small amount of bending displacement whereas the third one is more or less a pure torsional mode. The frequencies and modes shown in Fig. 2 agree completely with those obtained from the dynamic stiffness approach of Ref. [3]. These three modes were used (and subsequently found to be adequate) to compute the flutter speed and flutter frequency of the wing. Representative results are shown in Table 1 for three different ply angles. These results agreed completely with the ones obtained numerically by using an established program called CALFUN [3-5] which uses exact dynamic stiffness theory for composite wings.

4.0 CONCLUSIONS

An analytical method of flutter analysis of a uniform cantilever composite wing has been presented by deriving in explicit form each term needed for the flutter analysis (which-*hitherto*-has had to be treated numerically). This involved extensive symbolic computation to finally obtain expressions for generalised mass, generalised stiffness and generalised unsteady aerodynamic terms. The process was assisted greatly by the symbolic (algebraic) computation package REDUCE. The correctness and accuracy of the method is validated by numerical results obtained from an existing procedure. The proposed method is free from ill-conditioning problems usually associated with numerical matrix manipulation, and hence it can be used to solve bench-mark problems as an aid to validate approximate methods. The method offers prospects for aeroelastic developments in an optimisation environment.

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APPENDIX

Explicit integral expressions using REDUCE

The integrals used in the theoretical derivations of generalised mass, stiffness and aerodynamic terms can be classified under three general forms which are

$$\int_0^1 H_n^2(\xi)d\xi \quad \text{(A1)}$$

$$\int_0^1 H_n(\xi)\,\Psi_n(\xi)d\xi \quad \text{(A2)}$$

and

$$\int_0^1 H_n(\xi)\,\Psi_m(\xi)d\xi \quad \text{(A3)}$$

where $H_n(\xi)$ and $\Psi_m(\xi)$ are respectively bending and torsional modes corresponding to the n-th and m-th natural frequencies as given below.

$$H_n(\xi)=A_n\cosh\alpha_n\xi+B_n\sinh\alpha_n\xi+ C_n\cos\beta_n\xi+D_n\sin\beta_n+E_n\cos\gamma_n\xi+F_n\sin\gamma_n\xi \quad \text{(A4)}$$

$$\Psi_m(\xi)=P_m\cosh\alpha_m\xi+Q_m\sinh\alpha_m\xi+ R_m\cos\beta_m\xi+S_m\sin\beta_m+T_m\cos\gamma_m\xi+U_m\sin\gamma_m\xi \quad \text{(A5)}$$

Note that $H_m(\xi)$ and $\Psi_n(\xi)$ can be written by replacing the suffices n and m in Equations (A4) and (A5) by m and n respectively.

The analytical expressions for the above three integrals were obtained using REDUCE and manipulating the algebra very considerably.

The integral of Equation (A1) in explicit form is given by

$$\int_0^1 H_n^2(\xi)d\xi = \mu_n+\nu_n+\rho_n+\tau_n+\sigma_n+\lambda_n+\varepsilon_n+\zeta_n+\theta_n \quad \text{(A6)}$$

where

$$\mu_n=(A_n^2-B_n^2+C_n^2+D_n^2+E_n^2+F_n^2)/2 \quad \text{(A7)}$$

$$\nu_n=\{\beta_n\gamma_n(A_n^2+B_n^2)\sinh2\alpha_n +\alpha_n\gamma_n(C_n^2-D_n^2)\sin2\beta_n+$$

$$\alpha_n\beta_n (E_n^2-F_n^2)\sin2\gamma_n\}/(4\alpha_n\beta_n\gamma_n) \quad \text{(A8)}$$

$$\rho_n=-2C_n(\eta_{\gamma n}\gamma_n\cos\beta_n-\xi_{\gamma n}\beta_n\sin\beta_n+F_n\gamma_n)/(\beta_n^2-\gamma_n^2) \quad \text{(A9)}$$

$$\tau_n= 2C_n(\eta_{\alpha n}\alpha_n\cos\beta_n+\xi_{\alpha n}\beta_n\sin\beta_n-B_n\alpha_n)/(\alpha_n^2+\beta_n^2) \quad \text{(A10)}$$

$$\sigma_n= 2A_n(\eta_{\gamma n}\gamma_n\cosh\alpha_n+\xi_{\gamma n}\alpha_n\sinh\alpha_n+F_n\gamma_n)/(\alpha_n^2+\gamma_n^2) \quad \text{(A11)}$$

$$\lambda_n=-2D_n(\eta_{\gamma n}\gamma_n\sin\beta_n+\xi_{\gamma n}\beta_n\cos\beta_n-E_n\beta_n)/(\beta_n^2-\gamma_n^2) \quad \text{(A12)}$$

$$\varepsilon_n= 2B_n(\eta_{\gamma n}\gamma_n\sinh\alpha_n+\xi_{\gamma n}\alpha_n\cosh\alpha_n-E_n\alpha_n)/(\alpha_n^2+\gamma_n^2) \quad \text{(A13)}$$

$$\zeta_n= 2D_n(\eta_{\alpha n}\alpha_n\sin\beta_n-\xi_{\alpha n}\beta_n\cos\beta_n+A_n\beta_n)/(\alpha_n^2+\beta_n^2) \quad \text{(A14)}$$

$$\theta_n=(A_nB_n\beta_n\gamma_n\sinh^2\alpha_n+C_nD_n\alpha_n\gamma_n\sin^2\beta_n+$$

$$E_nF_n\alpha_n\beta_n\sin^2\gamma_n)/(\alpha_n\beta_n\gamma_n) \qquad (A15)$$

with

$$\eta_{\alpha n}=A_n\sinh\alpha_n+B_n\cosh\alpha_n\,,\ \eta_{\gamma n}=E_n\sin\gamma_n-F_n\cos\gamma_n \qquad (A16)$$

$$\xi_{\alpha n}=A_n\cosh\alpha_n+B_n\sinh\alpha_n\,,\ \xi_{\gamma n}=E_n\cos\gamma_n+F_n\sin\gamma_n \qquad (A17)$$

The integral (A2) in explicit form is given by

$$\int_0^1 H_n(\xi)\,\Psi_n(\xi)d\xi = \bar{\mu}_n+\bar{\nu}_n+\bar{\rho}_n+\bar{\tau}_n+\bar{\sigma}_n+\bar{\lambda}_n+\bar{\varepsilon}_n+\bar{\zeta}_n+\bar{\theta}_n \qquad (A18)$$

where

$$\bar{\mu}_n=\{k_\alpha(A_n^2-B_n^2)+k_\beta(C_n^2+D_n^2)+k_\gamma(E_n^2+F_n^2)\}/2 \qquad (A19)$$

$$\bar{\nu}_n=\{\beta_n\gamma_nk_\alpha(A_n^2+B_n^2)\sinh2\alpha_n+\alpha_n\gamma_nk_\beta(C_n^2-D_n^2)\sin2\beta_n+\alpha_n\beta_nk_\gamma(E_n^2-F_n^2)\sin2\gamma_n\}/(4\alpha_n\beta_n\gamma_n) \qquad (A20)$$

$$\bar{\rho}_n=-C_n(k_\beta+k_\gamma)(\eta_{\gamma n}\gamma_n\cos\beta_n-\xi_{\gamma n}\beta_n\sin\beta_n+F_n\gamma_n)/(\beta_n^2-\gamma_n^2) \qquad (A21)$$

$$\bar{\tau}_n=C_n(k_\alpha+k_\beta)(\eta_{\alpha n}\alpha_n\cos\beta_n+\xi_{\alpha n}\beta_n\sin\beta_n-B_n\alpha_n)/(\alpha_n^2+\beta_n^2) \qquad (A22)$$

$$\bar{\sigma}_n=A_n(k_\alpha+k_\gamma)(\eta_{\gamma n}\gamma_n\cosh\alpha_n+\xi_{\gamma n}\alpha_n\sinh\alpha_n+F_n\gamma_n)/\ (\alpha_n^2+\gamma_n^2) \qquad (A23)$$

$$\bar{\lambda}_n=-D_n(k_\beta+k_\gamma)(\eta_{\gamma n}\gamma_n\sin\beta_n+\xi_{\gamma n}\beta_n\cos\beta_n-E_n\beta_n)/(\beta_n^2-\gamma_n^2) \qquad (A24)$$

$$\bar{\varepsilon}_n=B_n(k_\alpha+k_\gamma)(\eta_{\gamma n}\gamma_n\sinh\alpha_n+\xi_{\gamma n}\alpha_n\cosh\alpha_n-E_n\alpha_n)/(\alpha_n^2+\gamma_n^2) \qquad (A25)$$

$$\bar{\zeta}_n=D_n(k_\alpha+k_\beta)(\eta_{\alpha n}\alpha_n\sin\beta_n-\xi_{\alpha n}\beta_n\cos\beta_n+A_n\beta_n)/(\alpha_n^2+\beta_n^2) \qquad (A26)$$

$$\bar{\theta}_n=(A_nB_nk_\alpha\beta_n\gamma_n\sinh^2\alpha_n+C_nD_nk_\beta\alpha_n\gamma_n\sin^2\beta_n+E_nF_nk_\gamma\alpha_n\beta_n\sin^2\gamma_n)/(\alpha_n\beta_n\gamma_n) \qquad (A27)$$

with $\eta_{\alpha n}$, $\eta_{\gamma n}$, $\xi_{\alpha n}$ and $\xi_{\gamma n}$ already defined in Equations (A6)-(A7)

The integral of Equation (A3) in explicit form is given as

$$\int_0^1 H_n(\xi)\Psi_m(\xi)=\mu_{mn}\cos\beta_m+\nu_{mn}\sin\beta_m+\rho_{mn}\cos\gamma_m+\tau_{mn}\sin\gamma_m+\sigma_{mn}\cosh\alpha_m+\lambda_{mn}\sinh\alpha_m+\varepsilon_{mn}+\zeta_{mn}+\theta_{mn} \qquad (A28)$$

where

$$\mu_{mn}=-(\eta_{\beta n}\beta_nR_m+\xi_{\beta n}\beta_mS_m)/\varphi_{mn}-(\eta_{\gamma n}\gamma_nR_m+\xi_{\gamma n}\beta_mS_m)/\kappa_{mn}+(\eta_{\alpha n}\alpha_nR_m-\xi_{\alpha n}\beta_mS_m)/\psi_{nm} \qquad (A29)$$

$$\nu_{mn}==-(\eta_{\beta n}\beta_nS_m-\xi_{\beta n}\beta_mR_m)/\varphi_{mn}-(\eta_{\gamma n}\gamma_nS_m-\xi_{\gamma n}\beta_mR_m)/\kappa_{mn}+(\eta_{\alpha n}\alpha_nS_m+\xi_{\alpha n}\beta_mR_m)/\psi_{nm} \qquad (A30)$$

$$\rho_{mn}==(\eta_{\beta n}\beta_nT_m+\xi_{\beta n}\gamma_mU_m)/\kappa_{nm}-(\eta_{\gamma n}\gamma_nT_m+\xi_{\gamma n}\gamma_mU_m)/\Delta_{mn}+(\eta_{\alpha n}\alpha_nT_m-\xi_{\alpha n}\gamma_mU_m)/\Omega_{nm} \qquad (A31)$$

$$\tau_{mn}==(\eta_{\beta n}\beta_nU_m-\xi_{\beta n}\gamma_mT_m)/\kappa_{nm}-(\eta_{\gamma n}\gamma_nU_m-\xi_{\gamma n}\gamma_mT_m)/\Delta_{mn}+(\eta_{\alpha n}\alpha_nU_m+\xi_{\alpha n}\gamma_mT_m)/\Omega_{nm} \qquad (A32)$$

$$\sigma_{mn}==(\eta_{\beta n}\beta_nP_m+\xi_{\beta n}\alpha_mQ_m)/\psi_{mn}+(\eta_{\gamma n}\gamma_nP_m+\xi_{\gamma n}\alpha_mQ_m)/\Omega_{mn}-(\eta_{\alpha n}\alpha_nP_m-\xi_{\alpha n}\alpha_mQ_m)/\delta_{mn} \qquad (A33)$$

$$\lambda_{mn}=(\eta_{\beta n}\beta_nQ_m+\xi_{\beta n}\alpha_mP_m)/\psi_{mn}+(\eta_{\gamma n}\gamma_nQ_m+\xi_{\gamma n}\alpha_mP_m)/\Omega_{mn}-(\eta_{\alpha n}\alpha_nQ_m-\xi_{\alpha n}\alpha_mP_m)/\delta_{mn} \qquad (A34)$$

$$\varepsilon_{mn}=-(\beta_nD_nR_m-\beta_mC_nS_m)/\phi_{mn}-(\gamma_nF_nR_m-\beta_mE_nS_m)/\kappa_{mn}-(\alpha_nB_nR_m-\beta_mA_nS_m)/\psi_{nm} \qquad (A35)$$

$$\zeta_{mn}=(\beta_nD_nT_m-\gamma_mC_nU_m)/\kappa_{nm}-(\gamma_nF_nT_m-\gamma_mE_nU_m)/\Delta_{mn}-(\alpha_nB_nT_m-\gamma_mA_nU_m)/\Omega_{nm} \qquad (A36)$$

$$\theta_{mn}=(\beta_nD_nP_m-\alpha_mC_nQ_m)/\psi_{mn}+(\gamma_nF_nP_m-\alpha_mE_nQ_m)/\Omega_{mn}+(\alpha_nB_nP_m-\alpha_mA_nQ_m)/\delta_{mn} \qquad (A37)$$

with $\eta_{\alpha n}$, $\eta_{\gamma n}$, $\xi_{\alpha n}$ and $\xi_{\gamma n}$ already defined in Equations (A6)-(A7) and

$$\eta_{\beta n}=C_n\sin\beta_n-D_n\cos\beta_n,\ \xi_{\beta n}=C_n\cos\beta_n+D_n\sin\beta_n \qquad (A38)$$

and

$$\psi_{mn}=\alpha_m^2+\beta_n^2,\ \kappa_{mn}^2=\beta_m^2-\gamma_n^2,\ \Omega_{mn}=\alpha_m^2+\gamma_n^2 \qquad (A39)$$

$$\psi_{nm}=\alpha_n^2+\beta_m^2,\ \kappa_{nm}^2=\beta_n^2-\gamma_m^2,\ \Omega_{nm}=\alpha_n^2+\gamma_m^2 \qquad (A40)$$

$$\varphi_{mn}=\beta_m^2-\beta_n^2,\ \Delta_{mn}=\gamma_m^2-\gamma_n^2,\ \delta_{mn}=\alpha_m^2-\alpha_n^2 \qquad (A41)$$

The above integrals have been checked numerically up to machine accuracy using Simpson's rule with five hundred equally spaced ordinates.

Table 1. Flutter speed and flutter frequency of a laminated composite wing.

Ply angle (θ) (deg)	Flutter speed (m/s)	Flutter frequency (rad/s)
-5	35.0	138.0
-8	38.8	150.0
-25	59.5	189.4

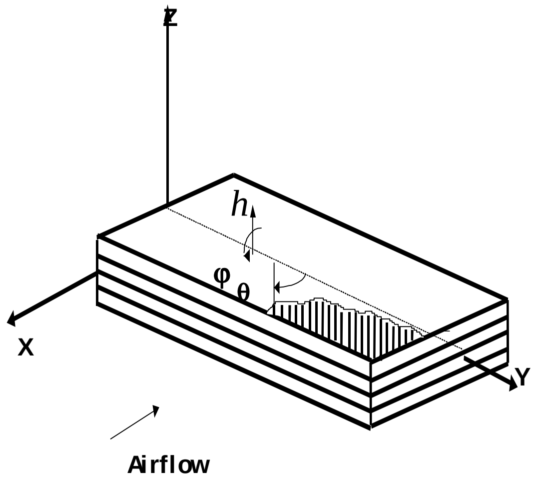

Fig. 1. Coordinate system and notation for a composite wing idealised as a beam element.

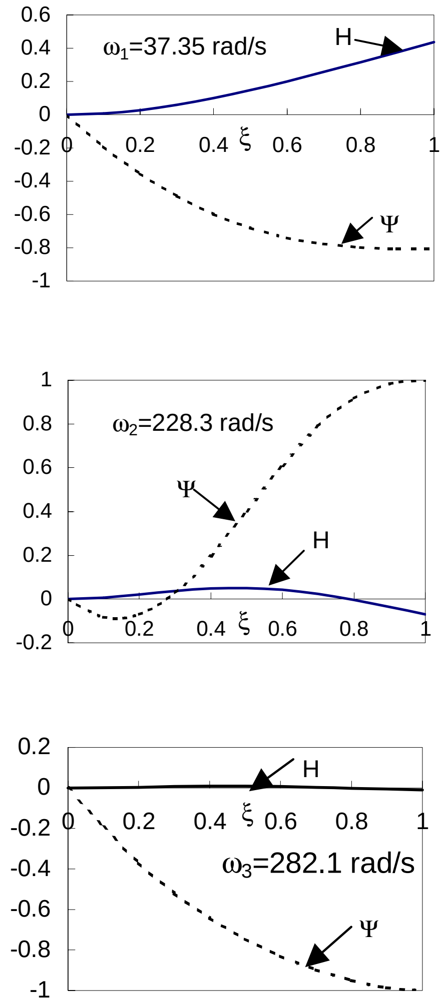

Fig. 2 Natural frequencies and mode shapes of a composite wing with $[-8^0]_{14}$ layup