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AN ANALYTICAL APPROACH TO HIGH-ORDER, NON LINEAR AIRPLANE STABILITY PROBLEMS

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Abstract

The longitudinal stability of a symmetric aircraft, is exhamined beyond the well known linearization about a steady mean state, leading to the phugoid and short-period modes. The exact equations of longitudinal motion of a symmetric aircraft are considered, i.e. balance of longitudinal and transverse force (without side force) and balance of pitching moment; the drag terms included are friction and lift-induced drag, plus non-symmetric lift-drag polar; the mass density is taken as a constant, as well as thrust along the flight path. Elimination would lead to a fourth-order non-linear differential equation for the angle-of-attack relative to the angle of zero pitching moment, if the acceleration of flight path angle is neglected, and the flight path angle is moderate. In the case of small flight path angle, it simplifies to a third-order differential equation, containing a set of non-linear corrections, to a second-order linear equation, specifying sinusoidal oscillation of the relative angle-of-attack. It is shown, by a small perturbation method, that forced oscillations occur at its harmonics (viz. double or triple frequency), and free oscillations can have decaying or growing amplitude. In the case of statically stable aircraft, the oscillations have a short period and grow or decay slowly. In the case of statically unstable aircraft the growth is rapid, as in a PIO.

List of Symbols

a	Constant parameter defined by eq. (8b)
A	Amplitude of free zero-order oscillation
b	Thrust-to-mass ratio, eq. (13a)
$\bar{b}$	Wing span (m)
B	Amplitude of free first-order oscillations, eq. (44a)
c	Mean aerodynamical chord (m)
C_1, C_2, C_3	Amplitudes of forced first-order oscillations, eq. (39)
$C_\pm, C_*$	Amplitudes of free first-order oscillations, eq. (45)
C_D	Drag coefficient
C_{D0}	Drag coefficient at zero angle of attack
$C_{D\alpha}$	Slope of drag coefficient
C_{Df}	Friction drag coefficient
C_L	Lift coefficient
C_{L0}	Lift coefficient at zero angle-of-attack
$C_{L\alpha}$	Slope of lift coefficient (rad $^{-1}$)
C_M	Pitching moment coefficient
C_{M0}	Pitching moment coefficient at zero angle-of-attack
$C_{M\alpha}$	Slope of pitching moment coefficient
D	Drag Force (kN)
f	Constant parameter defined by eq. (21a), (m^{-1})
$\bar{f}$	Constant parameter defined by eq. (13b), (m^{-1})
f_0	Parameter in thrust as function of airspeed, eq. (14a)
F	Non-linear return force, eq.(34)
g	Acceleration of gravity (9.81m s^{-2})
h	Constant parameter defined by eq. (21b)
$\bar{h}$	Constant parameter defined by eq. (13c)
I	Transverse moment of inertia (Kg m^2)
k	Coefficient of lift-induced drag, eq. (5)
L	Lift force
m	Mass (Kg)
M	Pitching moment
N	Number of cycles to double or halve amplitude
r	Constant parameter defined by eq. (31)
S	Reference or wing area (m^2)
t	Time
T	Thrust (kN)
U	airspeed (m/s)

W	Weight
α	Angle-of-attack (rad)
α_0	Angle-of-attack for zero lift (rad), eq. (4b)
α_1	Angle-of-attack for zero pitching moment (rad), eq. (6b)
γ	Flight path angle (rad)
ε	Constant parameter defined by eq. (21c)
$\bar{\varepsilon}$	Constant parameter defined by eq. (13d)
φ	Angle-of-attack relative to angle-of-attack for zero pitching moment (rad), eq. (18b)
φ_0	Relative angle-of-attack for fundamental oscillation (rad), eq. (32a)
φ_1	Relative angle-of-attack for first-order perturbation (rad), eq. (36a)
$\bar{\varphi}_1$	Relative angle-of-attack for free, first-order oscillation, eq. (44a)
λ	Coefficient in non-parabolic lift-drag polar, eq. (5)
μ	Constant non-linearity parameter, eq.(35)
ω	Frequency of oscillation (rad s^{-1}), eq. (36b)
ω_0	Frequency of fundamental oscillation (rad s^{-1}), eq. (29)
ω_1	Perturbation of frequency of oscillation, eq. (36b)
ω_*	Frequency of angle-of-attack oscillations, eq. (28b)
ρ	Mass density (Kg m^{-3})
τ	Time to double or halve amplitude (s)
τ_0	Period of oscillation (s)
θ	Angle-of-attack relative to angle-of-attack for zero lift (rad), eq. (9a)
θ_1	Angle-of-attack for zero pitching moment relative to angle-of-attack for zero lift (rad), eq. (16b)
υ	Growth or decay rate of oscillations (s^{-1}), eq. (46)

Introduction

The equations of the longitudinal motion of a symmetric aeroplane, are usually solved in the literature [1-10], using the method [11] of linearization of the fourth-order system about a mean state of steady flight, to obtain the frequency and damping of the phugoid and short period modes. An even older approach [12] , is to solve exactly the non-linear equations of motion, as shown by the phugoid [13]. The full system of non-linear equations of longitudinal stability is quite complex, and its solution has been approached by the method of bifurcations, together with numerical methods, viz. in the investigation of spins [13-20]. Explicit analytic solutions are rare, e.g. one example is the inverse phugoid problem, concernig the non-linear perturbation of flight along a constant glide slope, [21-24]; there is flight test data support the latter theory of non-linear stability with glide slope constraint [25] as there is for parameter identification methods [26-29]; this case involves only one degree-of-freedom, and and can be extended to include atmospheric effects [30-33]. In the present paper, the problem of non-linear stability with three degrees-of-freedom is addressed directly, by elimination between the fourth-order system of equations of motion, to obtain a single higher-order equation for the angle-of-attack. The approach is thus quite different from the phugoid, in the assumptions made, viz.: (i) the phugoid assumes flight at constant angle-of-attack, with drag equal to thrust, and unrestricted flight path angle, e.g. the loop is included; (ii) the present analysis restricts the flight path angle, and concentrates on the dynamics of the angle-of-attack, without requiring drag to balance thrust. Thus the present non-linear analysis of longitudinal aircraft stability is, in a sense, complementary to the theory of the phugoid.

The exact balance of lift, drag, weight, thrust and inertia force is taken, together with the pitching moment equation, under the assumptions of flight at low Mach number, away from the stall, for constant air density, and constant thrust along the flight path . This leads to a fourth-order system of coupled non-linear differential equations, which can be written using time as the independent variable; the dependent variables are the flight path angle, angle of attack and airspeed. Elimination between these equations, with neglect of the acceleration of flight path angle, leads (Case I) to a rather complicated non-linear, fourth-order differential equation for angle-of-attack. By restricting the flight path angle γ to (Case II) moderate values $\gamma \leq 30^0$ (so that $\gamma^2 \ll 1$ in radians), the flight path angle equation decouples, but the fourth-order non-linear differential equation for angle-of-attack, still remains fairly complicated . It is simplified further for (Case III) small flight path angle $\gamma \leq 4°\text{-}10°$ (or $\gamma \ll 1$ in radians), when it becames a third-order differential equation for angle-of-attack, consisting of: (i) linear terms, forming a second-order differential equation, with constant fundamental frequency, independent of airspeed, as for the short-period mode; (ii) one set of non-linear terms, associated with non-parabolic lift drag polar and lift-induced drag, which introduce respectively quadratic and cubic powers of the angle of attack; (iii) the other set of non-linear terms consists of products of powers of derivatives of angle-of-attack, which arise from the elimination among the equations of motion, and thus could be loosely interpreted as a coupling to the phugoid motion.

The method of solution which is used for case III, and would apply as well (with more tedious algebra) to cases I and II, is to consider a perturbation of the simple fundamental mode, with constant amplitude A and frequency ω_0. It is found that the non-linear effects do not cause a change in oscillation frequency, and induce oscillations at multiples the fundamental frequency. The calculation, to the lowest order in the perturbation parameter, of the amplitude and phase of the oscillation induced at the harmonics of the short-period frequency, can be made by perturbing the fundamental mode, and linearizing the third-order differential equation for angle-of-attack. The forced solution then specifies an excitation at the double and triple of the fundamental frequency, whereas the free oscillations lead to a cubic equation for the frequency ω. The roots of this cubic equation are, besides the fundamental period, a complex frequency whose imaginary part specifies an amplified or damped mode. The fundamental frequency is short, and the number of cycles to double amplitude long, for statically stable aircraft, whether of fighter or transport type. In the case of statically unstable aircraft, the roles of frequency and damping are interchanged, and thus the time scales for instability are short, leading to a rapid growth of angle-of-attack oscillations, as in PIO's (Pilot-Induced Oscillations).

Balance of forces (without side force) and pitching moment.

Considering the longitudinal motion of a symmetric aircraft (Figure 1), under lift L, drag D, thrust T (along the flight path) and weight W, balancing the inertia force, and the forces [12, p. 540] along the tangent (1a) and normal (1b) to the flight paths can be written:

$$m\dot{U} = T - D - W\sin\gamma, \tag{1a}$$

$$mU\dot{\gamma} = L - W\cos\gamma, \tag{1b}$$

where m is the mass, and the acceleration is written in tangential $\dot{U}$ and centripetal $U\dot{\gamma}$ components, U is the velocity, γ the flight path angle, and dot denotes time derivative, e.g. it appears twice applied to angle of attack α in the pitching moment M equation:

$$\dot{f} \equiv \partial f / \partial t: \quad I(\ddot{\alpha} + \ddot{\gamma}) = M, \tag{2}$$

where I denotes the moment of inertia relative to an axis transverse to the plane of motion (defined by the tangent and binormal to the trajectory).

The system of equations of motion (1a, b;2) is of the fourth-order, and its couplings and non-linearities are specified not only by the dynamics of rigid bodies, but also by the aerodynamic laws, for lift L, drag D and pitching moment M:

$$L = \tfrac{1}{2}\rho S U^2 C_L(\alpha), \tag{3a}$$

$$D = \tfrac{1}{2}\rho S U^2 C_D(\alpha), \tag{3b}$$

$$M = \tfrac{1}{2}\rho c S U^2 C_M(\alpha), \tag{3c}$$

where ρ is the mass density, S the reference area and c the mean aerodynamic chord, all taken as constant (no change in aircraft configuration and small altitude excursions), and for flight at low Mach number the lift C_L, drag C_D and pitching moment C_M coefficients, depend only on angle-of-attack, viz., for flight away from the stall: (i) the lift coefficient is a linear function of angle-of-attack:

$$C_L(\alpha) = C_{L0} + \alpha\, C_{L\alpha} = C_{L\alpha}(\alpha - \alpha_0), \tag{4a}$$

with value C_{L0} at zero angle-of-attack, and slope $C_{L\alpha}$, thus vanishing at the angle α_0 of zero lift:

$$\alpha_0 \equiv -C_{L0} / C_{L\alpha}; \tag{4b}$$

(ii) the drag coefficient is due to friction drag C_{Df}, and lift-induced drag with coefficient k, and a non-parabolic lift-drag polar term is included with coefficient λ:

$$C_D(\alpha) = C_{D0} + \lambda\, C_L(\alpha) + k[C_L(\alpha)]^2; \tag{5}$$

(iii) the pitching moment coefficient is again a linear function af angle-of-attack:

$$C_M(\alpha) = C_{M0} + \alpha\, C_{M\alpha} = C_{M\alpha}(\alpha - \alpha_1), \tag{6a}$$

with value C_{M0} at zero angle-of-attack, and slope $C_{M\alpha}$, thus vanishing at the angle:

$$\alpha_1 \equiv -C_{M0} / C_{M\alpha}, \tag{6b}$$

of zero pitching moment.

Coupling of airspeed, angle-of-attack and flight path angle

Starting with the radial force balance (1b):

$$\dot{\gamma} + (g/U)\cos\gamma = L/mU = (\rho S/2m) C_{L\alpha}(\alpha - \alpha_0) U \tag{7}$$

use of (3a,4a) leads to:

$$\dot{\gamma} = -gU^{-1}\cos\gamma + aU\theta, \tag{8a}$$

where a is a constant:

$$a \equiv \rho S C_{L\alpha} / 2m, \tag{8b}$$

and θ is the angle-of-attack relative to the angle of zero lift:

$$\theta \equiv \alpha - \alpha_0, \tag{9a}$$

in terms of which the lift coefficient (4a) can be written:

$$C_L(\theta) = C_{L\alpha}\, \theta. \tag{9b}$$

Similarly, in the tangential force balance (1a):

$$\dot{U}+g\sin\gamma-T/m=-D/m=-(\rho S/2m)U^2C_D, \tag{10}$$

(3b,5,9b) can be used:

$$\dot{U}+g\sin\gamma-T/m=-(\rho S/2m)U^2\left(C_{D0}+\lambda C_{L\alpha}\theta+kC_{L\alpha}^2\theta^2\right), \tag{11}$$

to obtain:

$$\dot{U}=b-g\sin\gamma-\bar{f}\,U^2\left(1+\bar{h}\theta+\bar{\varepsilon}\theta^2\right), \tag{12}$$

where:

$$b\equiv T/m, \tag{13a}$$

$$\bar{f}\equiv\rho SC_{D0}/2m, \tag{13b}$$

$$\bar{h}\equiv\lambda C_{L\alpha}/C_{D0}, \tag{13c}$$

$$\bar{\varepsilon}\equiv kC_{L\alpha}{}^2/C_{DO}; \tag{13d}$$

the last two (13c,d) are dimensionless constants, (13b) is the ballistic coefficient associated with friction drag, and (13a) is the thrust per unit mass; the latter is assumed to be also constant, implying either no throttle movement, or slow response compared with other time scales of the problem. The analysis would remain valid for a thrust per unit mass dependence on airspeed like:

$$T(U)/m=b-f_0\,U^2, \tag{14a}$$

where the constant f_0 adds to (13b):

$$\bar{f}=f_0+\rho SC_{D0}/2m, \tag{14b}$$

and (13a) is unchanged, with b defined by (14a).

In the remaining equation of motion, for pitching moment (2c):

$$\ddot{\alpha}+\ddot{\gamma}=(\rho Sc/2I)U^2C_M(\alpha), \tag{15a}$$

can be used (3c, 6a):

$$C_M(\alpha)=C_{M\alpha}(\alpha-\alpha_1)=C_{M\alpha}((\alpha-\alpha_0)-(\alpha_1-\alpha_0)), \tag{15b}$$

and also the angle-of-attack relative to the angle-of-attack for zero lift (9a):

$$C_M(\theta)=C_{M\alpha}(\theta-\theta_1), \tag{16a}$$

which is distinct from the angle-of-attack for zero pitching moment:

$$\theta_1\equiv\alpha_1-\alpha_0; \tag{16b}$$

substitution of (9a,16a) in (15a) yields:

$$0=\ddot{\alpha}+\ddot{\gamma}+jU^2(\theta-\theta_1), \tag{17a}$$

where j is again a constant:

$$j\equiv-\rho cSC_{M\alpha}/2I. \tag{17b}$$

In the fourth-order non-linear coupled system of differential equations of motion (8a, 12, 17a), the airspeed U, flight path angle γ and angle-of-attack relative to the angle of zero lift θ, appear as functions of time.

Elimination for small or moderate flight path angle

Subsequently three aproximations will be considered, as concerns the flight path angle, viz. (i) $\ddot{\gamma}\ll\ddot{\alpha}$, (ii) $\gamma^2\ll1$ and (iii) $\gamma\ll T/W$, whose implications are discussed next. The assumption (i) of acceleration of flight path negligible relative to acceleration of angle-of-attack $\ddot{\gamma}\ll\ddot{\alpha}$ restricts the kind of flight manouevers allowed, e.g. it holds for: (a) a descent on a nearly constant glide slope, in which case some pitch activity may be required to keep the flight slope; (b) horizontal flight, e.g. accelerating or decelerating. It would not hold for other manouevers, like rapid climbs or descents. The assumption (ii) of moderate flight path angle $\gamma^2\ll1$, i.e. not exceeding $\gamma<30°$, excludes steep climbs or descents, e.g. a loop would be excluded. A more restrictive assumption is (iii) small flight path angle relative to the thrust-to-weight ratio $\gamma\ll T/W$. This assumption is not met by existing aircraft in all conditions, and thus is again a restriction on manouevering, as will be shown by considering two examples. For a modern high-agility fighter the thrust-to-weight ratio in air combat configuration may reach T/W~1 to 1.3; these air superiority fighters can climb vertically $\gamma=90°$, so that $\gamma=\pi/2=1.57$ radians and $\gamma\ll T/W$ is not met generally, but only for flight path angles corresponding to $\gamma\le0.15$ or $\gamma\le10°$. For a large jet transport T/W~0.2 to 0.5, and the assumption $\gamma\ll T/W$ is even more restrictive,viz. $\gamma\le0.04$ or $\gamma\le4°$. Thus the range of flight path angles is restricted, though less so for high-agility fighter case. In spite of this, it is still worthwhile to study non-linear effects, since the methods used extend from case (iii) to (ii) or (i), with greater analytical complexity.

In order to facilitate the elimination of the system of equations of motion, the pitching moment equation (17a) can be put in the simplest form:

$$\ddot{\gamma}\ll\ddot{\varphi}:\quad \ddot{\varphi}+jU^2\varphi=0, \tag{18a}$$

by introducing the angle-of-attack relative to the angle of zero pitching moment:

$$\varphi\equiv\theta-\theta_1=\alpha-\alpha_1, \tag{18b}$$

besides neglecting the acceleration of the flight path angle $\ddot{\gamma}\ll\ddot{\varphi}$ relative to the acceleration of the angle-of-attack; in terms of the angle-of-attack relative to the angle-of-attack for zero pitching moment, which will be called henceforth 'relative angle-of-attack', the force balance equations (8a,12) become:

$$\dot{\gamma}+gU^{-1}\cos\gamma=aU(\varphi+\theta_1), \tag{19a}$$

$$\dot{U}=b-g\sin\gamma-fU^2(1+h\varphi+\varepsilon\varphi^2), \tag{19b}$$

where:

$$f\left(1+h\varphi+\varepsilon\varphi^2\right)=\bar{f}\left(1+\bar{h}\theta+\bar{\varepsilon}\theta^2\right)= =\bar{f}\left\{1+\bar{h}\left(\varphi+\theta_1\right)+\bar{\varepsilon}\left(\varphi+\theta_1\right)^2\right\}, \tag{20}$$

so that the new constants f, h, ε are specified in terms of the old (13b, c, d) by:

$$f \equiv \bar{f}\left(1+\theta_1\bar{h}_1+\theta_1^2\bar{\varepsilon}\right), \tag{21a}$$

$$fh \equiv \bar{f}\left(\bar{h}+2\theta_1\bar{\varepsilon}\right), \tag{21b}$$

$$f\,\varepsilon \equiv \bar{f}\,\bar{\varepsilon}. \tag{21c}$$

By substitution of (13b, c, d) in (21a) it follows that f is given by:

$$f=\left(\rho S/2m\right)\left(C_{D0}+\lambda\, C_{L\alpha}\theta_1+k\, C_{L\alpha}^2\theta_1^2\right)= =\left(\rho S/2m\right)C_D\left(\theta_1\right), \tag{21d}$$

where (5) was used.

The elimination among the equations of motion (18a; 19a, b), for the relative angle-of-attack without any further restriction (case i) would lead to a non-linear fourth-order differential equation, which is rather complicated. The deduction is simpler case (ii) of moderate flight path angle $\gamma \le 30°$, but still leads to a fourth-order differential equation. If the flight path is small $\gamma \le 10°$, in the precise sense (case iii) of negligible compared to the thrust-to-weight ratio:

$$\gamma << b/g=T/mg=T/W:\ \dot{\gamma}=aU(\varphi+\theta_1)-g/U, \tag{22}$$

the flight path angle is specified by the radial force balance equation (22), which decouples from the other equations, viz. the tangential force balance and pitching moment equations:

$$\ddot{\varphi}=-jU^2\varphi, \tag{23a}$$

$$\dot{U}=b-fU^2\left(1+h\varphi+\varepsilon\varphi^2\right), \tag{23b}$$

which lead to a non-linear third-order differential equation for the angle-of-attack, as follows next.

Third-order non-linear differential equations for the relative angle-of-attack

In order to perform the elimination, of the system (23a, b) for small flight path angle it is sufficient to solve (23a) for U:

$$U\sqrt{j}=\pm i\ddot{\varphi}^{1/2}\varphi^{-1/2}, \tag{24a}$$

$$\mp 2i\dot{U}\sqrt{j}=\ddot{\varphi}^{-1/2}\,\varphi^{-1/2}\,\dddot{\varphi}-\varphi^{-3/2}\dot{\varphi}\ddot{\varphi}^{1/2}, \tag{24b}$$

and substitute in (23b), leading to a third-order non-linear differential equation for the relative angle-of-attack:

$$0=\dddot{\varphi}-\varphi^{-1}\dot{\varphi}\ddot{\varphi}\pm 2ib\sqrt{j}\,\varphi^{1/2}\ddot{\varphi}^{1/2}\pm 2i\left(f/\sqrt{j}\right) \left(1+h\varphi+\varepsilon\varphi^2\right)\varphi^{-1/2}\ddot{\varphi}^{3/2}; \tag{25}$$

once this is solved for the relative angle-of-attack $\varphi(t)$, the velocity U(t) and flight path angle $\gamma(t)$ are determined, respectively from (24a) and (22).

Non-sinusoidal oscillation with anharmonic and other non-linear terms

In order to interpret the differential equation for the relative angle-of-attack (25), it can be re-written in the form:

$$\dddot{\varphi}\left[1+h\varphi+\varepsilon\varphi^2\right]+\left(bj/f\right)\varphi= =\pm\left(i\sqrt{j}/2f\right)\left(\varphi^{1/2}\ddot{\varphi}^{-1/2}\dddot{\varphi}-\varphi^{-1/2}\dot{\varphi}\ddot{\varphi}^{1/2}\right), \tag{26}$$

From (2, 3c, 6a) it follows that:

$$\ddot{\gamma}<<\ddot{\alpha}:\ \ddot{\alpha}-\left(\rho cSC_{M\alpha}U^2/2I\right)\left(\alpha-\alpha_1\right)=0, \tag{27}$$

so that (18b):

$$\ddot{\varphi}+\omega_*^2\varphi=0, \tag{28a}$$

where ω_* is the frequency of the oscillation of the relative angle-of-attack for constant airspeed U:

$$\omega_*^2\equiv-\rho cSC_{M\alpha}U^2/2I=jU^2. \tag{28b}$$

In the present problem the airspeed is not constant, and the role of frequency of oscillation of the relative angle-of-attack or fundamental frequency is played by:

$$\omega_0^2\equiv bj/f=-\left(T/mf\right)\rho cSC_{M\alpha}/2I=-cTC_{M\alpha}/IC_D\left(\theta_1\right), \tag{29}$$

which corresponds to (28b) with the substitution $U^2 \leftrightarrow T/mf$ and can be simplified using (13a, 17b, 21d). The value of ω_0 is weakly dependent on airspeed as for the short period frequency.

Since the oscillation frequency concerns the relative angle-of-attack, and involves the rotational inertia I, it corresponds, loosely speaking, a kind of short-period mode; strictly speaking, this is not the short-period mode in the original sense, as the latter arises from linearization of the equations of motion. Therefore it may be better instead to designate ω_0 as fundamental frequency of the oscillation of relative angle-of-attack, henceforth refered to simply as the fundamental frequency. Substituting (29) in (26) it follows that the linear terms are similar to (28a), i.e. represent an oscillation with constant amplitude and frequency ω_0.

$$\dddot{\varphi}\left[1+h\varphi+\varepsilon\varphi^2\right]+\omega_0^2\varphi= =\pm ir\left(\varphi^{1/2}\ddot{\varphi}^{-1/2}\dddot{\varphi}-\varphi^{-1/2}\dot{\varphi}\ddot{\varphi}^{1/2}\right), \tag{30}$$

although this is modified by the non-linear terms, involving h, ε and:

$$r\equiv\sqrt{j}/2f=\left(1/2f\right)\sqrt{-\rho cSC_{M\alpha}/2I}= =\left[m/C_D\left(\theta_1\right)\right]\sqrt{-C_{M\alpha}c/2\rho SI}, \tag{31}$$

where it has been taken into account that static pitch stability requires $C_{M\,\alpha}<0$ and (17b, 21d) were used. The sinusoidal oscillation mode (28a) ressembles a linear or harmonic oscillator in classical mechanics, but in the present case (30), besides the linear part, there are two sets of non-linear terms. The first, in square brackets, resembles a non-linear or anharmonic oscillator, in that it involves powers of the oscillation variable, i.e. the relative angle-of-atack, but they multiply $\ddot\varphi$ rather than φ; they are due, from h, ε (21a, b, c), and $\bar f, \bar g, \bar h$ (13b, c, d) to the non-constant terms in the lift-drag polar, i.e. the lift-induced drag $\kappa\neq 0$ and non-parabolic $\lambda\neq 0$ lift-drag polar. The second set of non-linear terms, on the r.h.s., involve products of powers of derivatives of the oscillation variable, and the coefficient involves r (31) and is out of phase by $\pi/2$; since these terms arise from the elimination, for relative angle-of-attack, between the pitching moment and force equations, they represent, in a loose terminology, a sort of coupling between the 'short period' and 'phugoid modes'. Again we are not concerned with a true phugoid mode, in the original sense, since the restriction to small flight path angle, limits the phugoid motions possible. It may be more unambiguous to refer to the non-linear terms, as perturbations of the fundamental sinusoidal oscillation of relative angle-of-attack.

Perturbation expansion for relative angle-of-attack and frequency

The linearization of (30), viz:

$$\ddot\varphi_0+\omega_0^2\varphi_0=0, \tag{32a}$$

leads to an oscillation of relative angle-of-attack, with constant amplitude A and fundamental frequency ω_0:

$$\varphi_0(t)=A\exp(i\omega_0 t). \tag{32b}$$

The sinusoidal function also satisfies the non-linear terms in curved brackets in (30), viz.:

$$\varphi_0^{1/2}\,\ddot\varphi_0^{-1/2}\,\dddot\varphi_0=\varphi_0^{-1/2}\,\dot\varphi_0\,\ddot\varphi_0^{1/2}, \tag{32c}$$

and thus the non-linear differential equation:

$$\ddot\varphi_0+\omega_0^2\varphi_0=\pm ir\left(\varphi_0^{1/2}\ddot\varphi_0^{-1/2}\dddot\varphi_0-\varphi_0^{-1/2}\dot\varphi_0\ddot\varphi_0^{1/2}\right), \tag{33}$$

also has the simple solution (32b); however, this simple solution does not extend to (30), on account of the terms in square brackets. The latter act like a non-linear return force:

$$\ddot\varphi=-\omega_0^2\varphi/\left(1+h\varphi+\varepsilon\varphi^2\right)\equiv F(\varphi). \tag{34}$$

We may take ε as a parameter measuring the importance of non-linear effects:

$$\begin{aligned}\mu\cong h/\varepsilon:\ &\left[1+\varepsilon\varphi(\varphi+\mu)\right]\ddot\varphi+\omega^2\varphi=\\ &=\pm ir\left(\varphi^{1/2}\,\ddot\varphi^{-1/2}\,\dddot\varphi-\varphi^{-1/2}\,\dot\varphi\,\ddot\varphi^{1/2}\right),\end{aligned} \tag{35}$$

and μ accounts for a non-parabolic lift-drag polar. Note that if the lift-drag polar is parabolic $\mu=0$, and the term in square brackets in (35) simplifies to $1+\varepsilon\varphi^2$, involving the lift-induced drag. Thus the non-linearity parameter ε is defined from the lift-induced drag, and can be used whether the lift-drag polar is parabolic or not. The solution of (35) is sought as a perturbation expansion:

$$\varphi=\varphi_0+\varepsilon\,\varphi_1+O\left(\varepsilon^2\right), \tag{36a}$$

$$\omega=\omega_0+\varepsilon\,\omega_1+O\left(\varepsilon^2\right), \tag{36b}$$

both in the relative angle-of-attack (36a) and in the fundamental frequency (36b).

Induced oscillations at multiples of the fundamental frequency

When substituting (36a, b), the zeroth-order term, which is independent of ε, coincides with (33), and hence vanishes; the first-order term, i.e. the coefficient of ε is:

$$\begin{aligned}&\dddot\varphi_1+\left[\pm(i/r)\varphi_0^{-1/2}\ddot\varphi_0^{1/2}-(1/2)\left(\ddot\varphi_0^{-1}\dddot\varphi_0+\varphi_0^{-1}\dot\varphi_0\right)\right]\\ &\ddot\varphi_1-\varphi_0^{-1}\ddot\varphi_0\dot\varphi_1+\\ &\left[\pm(i/r)\omega_0^2\varphi_0^{-1/2}\ddot\varphi_0^{1/2}+(1/2)\varphi_0^{-1}\left(\dddot\varphi_0+\varphi_0^{-1}\dot\varphi_0\ddot\varphi_0\right)\right]\varphi_1\\ &=\mp(i/r)\left[2\omega_0\omega_1\varphi_0^{1/2}\ddot\varphi_0^{1/2}+\omega_0^2\varphi_0^{1/2}\ddot\varphi_0^{3/2}(\varphi_0+\mu)\right],\end{aligned} \tag{37}$$

which is a third-order differential equation for the perturbation, with (32b) appearing in the coefficients:

$$\begin{aligned}&\dddot\varphi_1-\omega_0(\pm 1/r+i)\ddot\varphi_1+\omega_0^2\dot\varphi_1-\omega_0^3(\pm 1/r-i)\varphi_1=\\ &=\pm(\omega_0^2/r)Ae^{i\omega_0 t}\left[2\omega_1-\omega_0 A^{i\omega_0 t}\left(\mu+Ae^{i\omega_0 t}\right)\right]\end{aligned} \tag{38}$$

Thus a forced solution is sought in the form of a superposition of oscillations, at the fundamental frequency plus two harmonics:

$$\varphi_1(t)=C_1\,e^{i\omega_0 t}+C_2\,e^{2i\omega_0 t}+C_3\,e^{3i\omega_0 t}; \tag{39}$$

on substitution of (39) into (38) yields:

$$O.C_1=2\omega_1\omega_0^2 A/r, \tag{40a}$$

$$A^2\mu=-3(1\mp ir)C_2, \tag{40b}$$

$$A^3=-8(1\mp ir)C_3. \tag{40c}$$

From (40a) it follows that $\omega_1=0$, so that there is no frequency shift $\omega=\omega_0$ in (36b), and no resonant term in (39), i.e. the first term is included in (32b) with C_1 being absorbed into the amplitude A ; note that there would be a frequency shift, for the anharmonic oscilator (30), if the non-linearities in curved brackets were absent. Thus the perturbation (39) is forced only at the first two harmonics of the fundamental:

$$\varphi_1(t) = -A^2 e^{2i\omega_0 t}\left[\mu/3 + (A/8)e^{i\omega_0 t}\right]/(1 \mp ir); \tag{41}$$

the real part is:

$$\varphi_1(t) = -\left[A^2/\left(1+r^2\right)\right]\left\{\mu/3\left[\cos(2\omega_0 t) \mp r\sin(2\omega_0 t)\right] + (A/8)\left[\cos(3\omega_0 t) \mp r\sin(3\omega_0 t)\right]\right\}, \tag{42}$$

and on substitution in (36a), together with (32b), specifies the complete oscillation.

Free relative angle-of-attack oscillation with damping or growth in time

Concerning the free oscillations of (38), viz.:

$$\dddot{\varphi}_1 - \omega_0(\pm 1/r + i)\ddot{\varphi}_1 + \omega_0^2\dot{\varphi}_1 - \omega_0^3(\pm 1/r + i)\varphi_1 = 0, \tag{43}$$

a solution is sought in the form of a sinusoidal oscillation:

$$\overline{\varphi}_1(t) = B e^{i\omega t}, \tag{44a}$$

where the frequency ω satisfies the cubic equation (44b):

$$\left(\omega^2 - \omega_0^2\right)\left[\omega + (-1 \pm i/r)\omega_0\right] = 0. \tag{44b}$$

As it could be expected from $\omega_1 = 0$ and $\omega = \omega_0$ in (40a, 36b), $\pm\omega_0$ are roots of (44b), and thus the third root is easily found:

$$\omega = \pm\omega_0, -\omega_0(1 + i/r): \ \overline{\varphi}_1(t) = C_+ e^{i\omega_0 t} + C_- e^{-i\omega_0 t} + C_* e^{i\omega_0 t} e^{\pm(\omega_0/r)t}, \tag{45}$$

i.e. the motion corresponds to oscillations at the fundamental frequency with constant amplitudes $C_\pm$, plus a term exponentially growing or decaying in time, in proportion to (29, 31):

$$\vartheta \equiv \omega_0 / r = 2\sqrt{bf} = 2\sqrt{fT/m} \equiv 0.693/\tau, \tag{46}$$

where τ defines the time scale for growth or decay by a factor of 2.

Time to double or halve amplitude

The present method of solution of the equations of longitudinal motion of a symmetric aeroplane, leaves in the elimination (24a, b), an uncertainty of sign, which persists through the first-order perturbation up to (45), allowing for modes growing or decaying amplitude, with time to double or halve amplitude τ given by (46) or, using (21d):

$$0.693/\tau \equiv \vartheta = \omega_0/r = 2\sqrt{bf} = \sqrt{2\rho STC_D(\theta_1)}/m. \tag{47}$$

Thus unstable modes can exist, and an estimate of their growth rate (47) is given by:

$$\vartheta = \sqrt{2\rho}\sqrt{T/m}\sqrt{S/m}\left(C_{D0} + \lambda C_{L\alpha}\theta_1 + kC_{L\alpha}^2\theta_1\right)^{1/2}. \tag{48}$$

In the latter the term in the third square root is larger for lower wing loading, which is a design feature of high manoueverability air superiority fighters; these also have a high thrust-to-weight ratio, so that the second square root is also larger for these aircraft; the growth rate of instabilities is large for high-drag coefficient (calculated for the angle θ_1 in the second curved brackets), e.g. in landing configuration or curved flight at high turn rate. The fundamental frequency ω_0 and period τ_0 are given by (29, 13a, 17b, 47):

$$\omega_0^2 = bj/f = -c(T/I)\left[C_{M\alpha}/C_D(\theta_1)\right] = (2\pi/\tau_0)^2 \tag{49}$$

and the number of periods to double or halve amplitude by:

$$N = \tau/\tau_0 = 0.110\, r = \left[m/C_D(\theta_1)\right]\sqrt{-c\,C_{M\alpha}/2\rho SI} \tag{50}$$

where the dimensionless quantity r was used (31). These quantities are calculated using data from [9, 34] in tables I for the F-4 and II for the B747.

Examples of fighter and transport aircraft

The calculation for a fighter (the McDonnell Douglas F-4 Phantom II) in Table I concerns three flight regimes, namely, approach to land and subsonic and supersonic cruise; the corresponding Mach number, true airspeed and altitude are indicated. From the slope $C_{M\alpha}$ and value at zero angle-of-attack C_{M0} of the pitching moment coefficient, follows (6b) the angle-of-attack for zero pitching moment α_1; in similar way, from the lift coefficient, follows (4b) the angle of zero lift α_0, and hence (16b) the angle θ_1 of zero pitching moment relative to the angle of zero lift. The drag coefficient at this angle, follows from the slope $C_{D\alpha}$ and value at zero angle of attack C_{D0} of the drag-coefficient:

$$C_D(\theta_1) = C_{D0} + C_{D\alpha}(\theta_1)^2. \tag{51}$$

Using this, the mass m of the aeroplane, the atmospheric mass density ρ and the wing area S, specifies the parameter f (21d), which has the dimensions of inverse length: $[f] = L^{-1}$. From the angle-of-attack α, and drag coefficient $C_D(\alpha)$, follows the drag (3b), which equals the thrust in steady, straight and level flight, viz. this will be exactly true for cruise, and approximately so for a stabilized approach with small flight path angle. The thrust T and the mass m specify b (13a) which has the dimensions of acceleration, $[b] = LT^{-2}$. The moment of inertia I is needed to calculate the parameter j (17b), which has the dimensions of inverse length squared $[j] = L^{-2}$ as follows from (18a). The three parameters f, b, j specify the fundamental frequency ω_0 and period τ_0 (49), and the growth rate ϑ and timescale τ (47, 48) and hence the number of periods to double amplitude N (50). The calculations for a large transport (Boeing 747) in Table II, concern again three flight regimes, viz. approach to land, and cruise at low or high altitude. The calculations are similar to Table I, with the simplification that the pitching moment is zero at zero angle-of-attack. In all cases the fundamental period is short, viz. 1-3 s for a fighter and longer 10-20 s for a transport; the growth time is much longer, viz. 30-170 s, so that the number of fundamental periods to double or halve amplitude is large, viz. N~3 to 12 for the transport and N~10 to 69 for the fighter, meaning that it is a slow growing instability, with plenty of time for compensation, even by manual control.

Example of a statically unstable aircraft

Consideration as been given so far to statically stable aircraft $C_{M\alpha} < 0$, for which (17b) the parameter j is positive $j > 0$, and thus the fundamental frequency (49) real. In the case of a statically unstable aircraft $C_{M\alpha} > 0$, then $j < 0$;

$$C_{M\alpha} > 0: \quad j = -|j|, \tag{52a}$$

and the fundamental frequency is imaginary:

$$\omega_0 = \sqrt{bj/f} = \pm i\sqrt{b|j|/f} = \pm i|\omega_0|, \tag{52b}$$

so that now the roles of ω_0 and ϑ are interchanged

$$\exp[\omega_0(i \pm 1/r)t] = \exp[\mp|\omega_0|t + i(|\omega_0|/r)t], \tag{53}$$

i.e. $\overline{\omega}_0 = |\omega_0|/r$ is the oscillation frequency and $\overline{\vartheta} = |\omega_0|$ the growth or decay rate. Taking as an example data [35] for the F-16C, which has a take-off mass m=12040 Kg, and afterburning thrust T=106.3 KN, corresponding to a thrust-to-weight ratio T/W=T/mg=0.90. The wing area is S=27.88m^2, and the span $\overline{b} = 9.45$m, corresponds to a mean chord $c = \overline{b}^2/S = 3.20$m. Taking $C_D(\theta_1) = 0.25$ as for the F-4, the oscillation frequency is $\overline{\omega}_0 = |\vartheta| = 0.460\,s^{-1}$, corresponding to a period $\overline{\tau}_0 = 2\pi/\overline{\omega}_0 = 137s$.

Using $\mu = |C_0(\theta_1)/C_{M\alpha}| = 4.4\text{x}10^5 \text{Kg}\,m^2$ as for the F-4, the growth rate $\overline{\vartheta} = |\omega_0| = 0.88\,s^{-1}$ and time to double amplitude $\overline{\tau} = 0.693/\overline{\vartheta} = 0.79s$ show that the instability is rapid.

The original analysis of the "phugoid" [13] uses [12] the force balance equations (1a,b), with constant thrust equal to drag and constant angle-of-attack:

$$m\dot{U} = -W \sin\gamma, \tag{54a}$$

$$mU\dot{\gamma} = L - W\cos\gamma. \tag{54b}$$

A complementary 'short-period' analysis would use the pitching moment equation (2; 3c; 6a):

$$\ddot{\gamma} \ll \ddot{\alpha}: \ \ddot{\alpha} - (C_{M\alpha}\,\rho c S/2I)\,U^2(\alpha - \alpha_1) = 0, \tag{55}$$

and also the condition of thrust equal to drag (3b):

$$T = D = \tfrac{1}{2}\rho S U^2\, C_D(\alpha). \tag{56}$$

The equation of angle-of-attack oscillations (55) is non-linear, because the airspeed U depends on angle-of-attack through (56); it is linearized by evaluating airspeed at the 'mean state', i.e. angle of attack of zero pitching moment:

$$U_1 = \sqrt{2T/\rho S C_{D\alpha}(\alpha_1 - \alpha_0)}. \tag{57}$$

Substituting in (56) in (55) yields:

$$\varphi = \alpha - \alpha_1: \ \ddot{\varphi} - [C_{M\alpha} cT/I C_D(\theta_1)]\varphi = 0, \tag{58}$$

which coincides with (32a, 29) and specifies: (i) a sinusoidal oscillation of frequency ω_0 for statically stable aircraft $C_{M\alpha}<0$; (ii) an amplitude growth or decay with time-to-double or halve of $0.693/|\omega_0|$, for a statically unstable aircraft $C_{M\alpha}>0$. The determination of the other parameter r (31) or ϑ (47, 48) requires a non-linear analysis [36], beyond the simple deduction (55-57), which complements the phugoid problem.

Conclusion

The last two decades have seen major progress in flight control technology, as witnessed by improvement in handling qualities of the current fighters (F-14 Tomcat, F-15 Eagle, F-16 Falcon, F-18 Hornet) compared with the century series (F-100 Super Sabre, F-101 Voodoo, F-102 Delta Dagger, F-104 Starfighter, F-105 Thunderchief, F-106 Delta Dart, F-111 Aardward and also F-8 Crusader and F-4 Phantom II). Staticaly unstable designs give improved manoueverability, and allow a smaller design for the same mission, and active control technology can provide gust alleviation and load limitation, as well as protection from departure at high angle-of-attack and/or sideslip, yielding an expanded carefree manouever envelope. Modern control technology also brought some partially unsolved problems, like the PIO (Pilot Induced Oscillation), which has caused accidents of both manned (F-22, Gripen) and unmanned (Darkstar) aircraft. Maybe the progress in flight control technology should be matched by advances in flight dynamics, into the unsteady and non-linear regimes, which have received less attention in the literature, but may hold the key to a better understanding, not only of spins, but perhaps also of PIOs.

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Legends for the Tables

Table I - Calculation of fundamental frequences and growth rate for F-4.
Table II - As Table I for Boeing 747.

Legend for the Figure

Figure 1- Balance of forces on the osculating plane of the flight path, and of pitching moment orthogonal to this plane.

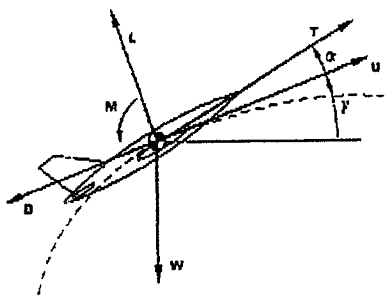

FIGURE 1

TABLE I

McDonnell Douglas F-4 Phantom II
c=4.88 m   S=49.2 m^2

		Cruise		
Condition	Approach to land	subsonic	supersonic	Units
M	0.206	0.900	1.800	-
z	0	35	55	x10^{3}ft
U	70.1	267	531	ms^{-1}
C_{M0}	+0.020	+0.025	-0.025	-
$C_{M\alpha}$	-0.098	-0.400	-0.780	rad^{-1}
α_I	+0.204	+0.0625	-0.032	rad
C_{L0}	0.430	0.100	0.010	-
$C_{L\alpha}$	2.80	3.75	2.80	rad^{-1}
α_0	-0.154	-0.0267	-0.00357	rad
θ_1	+0.358	0.0892	-0.0284	rad
C_{D0}	0.0269	0.0205	0.0439	-
$C_{D\alpha}$	0.555	0.300	0.400	rad^{-1}
$C_D(\theta_1)$	0.226	0.0285	0.0325	-
m	15060	17690	17690	Kg
ρ	1.293	0.400	0.143	Kg m^{-3}
f	4.77x10^{-4}	1.58x10^{-5}	6.46x10^{-6}	m^{-1}
α	11.7	2.6	3.3	0
α	0.204	0.0454	0.0576	rad
$C_D(\alpha)$	0.050	0.0211	0.0452	-
D=T	7.815	14.8	44.8	kN
b	0.519	0.837	2.534	ms^{-2}
I	1.913	2.00	2.00	x10^5 Kg m^2
j	7.95x10^{-5}	9.60x10^{-5}	6.70x10^{-5}	m^{-2}
ω_0	0.294	2.25	5.13	5^{-1}
τ_0	2.14	2.79	1.23	s
ϑ	3.15x10^{-2}	7.27x10^{-3}	8.09x10^{-3}	s^{-1}
τ	22.0	94.9	85.9	s
N	10.3	34.0	69.4	N

TABLE II

Boeing 747

c=8.32 m

S=511 m^2

Condition	Approach to land	Cruise at low	Cruise at high	Altitude
M	0.198	0.650	0.900	
z	0	20	40	x10^3 ft
U	67.4	205	265	ms^{-1}
C_{M0}	0	0	0	-
$C_{M\alpha}$	-1.45	-1.00	-1.60	rad^{-1}
α_1	0	0	0	rad
C_{L0}	0.92	0.21	0.29	-
$C_{L\alpha}$	5.67	4.4	5.5	rad^{-1}
α_0	-0.162	-0.0477	-0.0527	rad
θ_1	+0.162	+0.0477	+0.0527	rad
C_{D0}	0.0269	0.0205	0.0439	-
$C_{D\alpha}$	0.555	0.300	0.400	rad^{-1}
$C_D(\theta_1)$	0.117	0.0348	0.0650	-
m	255830	288778	288778	Kg
ρ	1.293	0.689	0.318	Kg m^{-3}
f	1.51x10^{-4}	2.12x10^{-5}	1.82x10^{-5}	m-1
α	8.5 0.148	2.5 0.044	2.4 0.042	0 rad
$C_D(\alpha)$	0.100	0.0168	0.313	-
D=T	150	124	178	kN
b	0.586	0.429	0.616	m^{-2}
I	6.17	7.11	7.11	x10^7 Kg m^2
j	6.46x10^{-5}	2.06x10^{-5}	1.51x10^{-5}	m^{-2}
ω_0	0.500	0.646	0.715	s^{-1}
τ_0	12.5	9.73	8.79	s
ϑ	1.88x10^{-2}	6.03x10^{-3}	6.70x10^{-3}	s^{-1}
τ	36.7	115	103	s
N	2.94	11.8	11.7	-